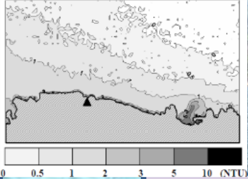
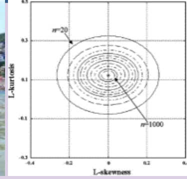
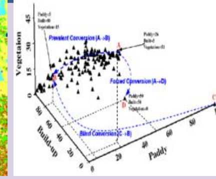
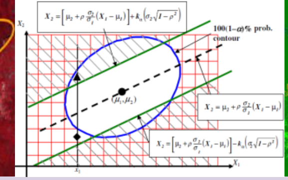
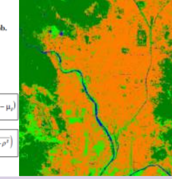
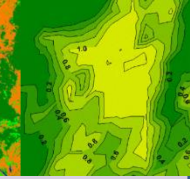






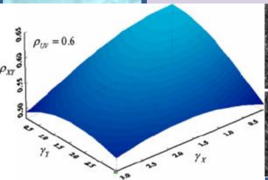
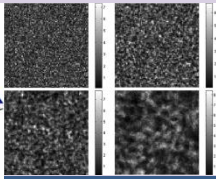
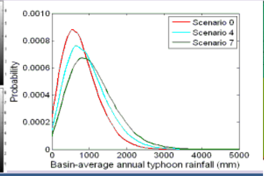




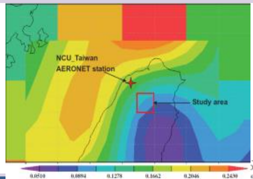
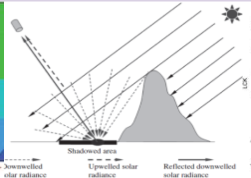
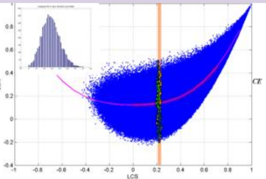
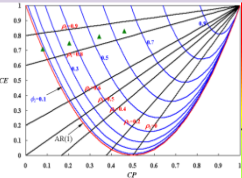
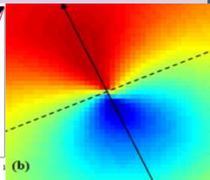
Experiences of Hydrological Frequency Analysis in Taiwan

- Coping with Extraordinary Extremes and Climate Change

Prof. Ke-Sheng Cheng

Department of Bioenvironmental Systems Engineering, National Taiwan University


Outline

- **Introduction**
- **Improving accuracy of frequency analysis**
  - Presence of outliers (extraordinary rainfall extremes)
  - Spatial correlation (non-Gaussian random field simulation)
- **Frequency analysis of multi-site rainfall extremes**

Introduction

- **Frequent occurrences of disasters induced by heavy rainfalls in Taiwan**
  – Landslides
  – Debris flows
  – Flooding and urban inundation
- **Increasing occurrences of rainfall extremes (some of them are record breaking) in recent years**
- **Almost all of the long-duration rainfall extremes (longer than 12 hours) were produced by typhoons.**

Examples of annual max events in Taiwan

Design durations

1965-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	9/5	8/18	8/18	8/18	8/18	8/18	8/18	8/18	8/18	8/18
Wutuh	6/11	8/21	9/6	9/6	9/6	9/6	8/18	8/18	8/18	8/18
1969-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	5/14	5/14	5/14	5/14	9/26	9/26	9/9	9/9	9/9	9/9
Wutuh	9/21	9/8	9/8	9/8	9/8	9/8	9/8	9/8	9/8	9/8
1974-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	9/27	9/27	10/11	10/11	10/11	10/11	10/11	10/11	10/11	10/11
Wutuh	9/15	9/15	10/11	10/11	9/15	10/11	10/11	10/11	10/11	10/11

1983-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	5/23	5/23	5/23	5/23	5/23	10/10	10/10	10/10	10/10	10/10
Wutuh	10/1	10/1	6/3	6/3	6/3	10/1	10/1	10/1	10/1	10/1

1987-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	7/26	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22
Wutuh	10/22	7/26	10/22	10/22	10/22	10/22	10/22	10/22	10/22	10/22

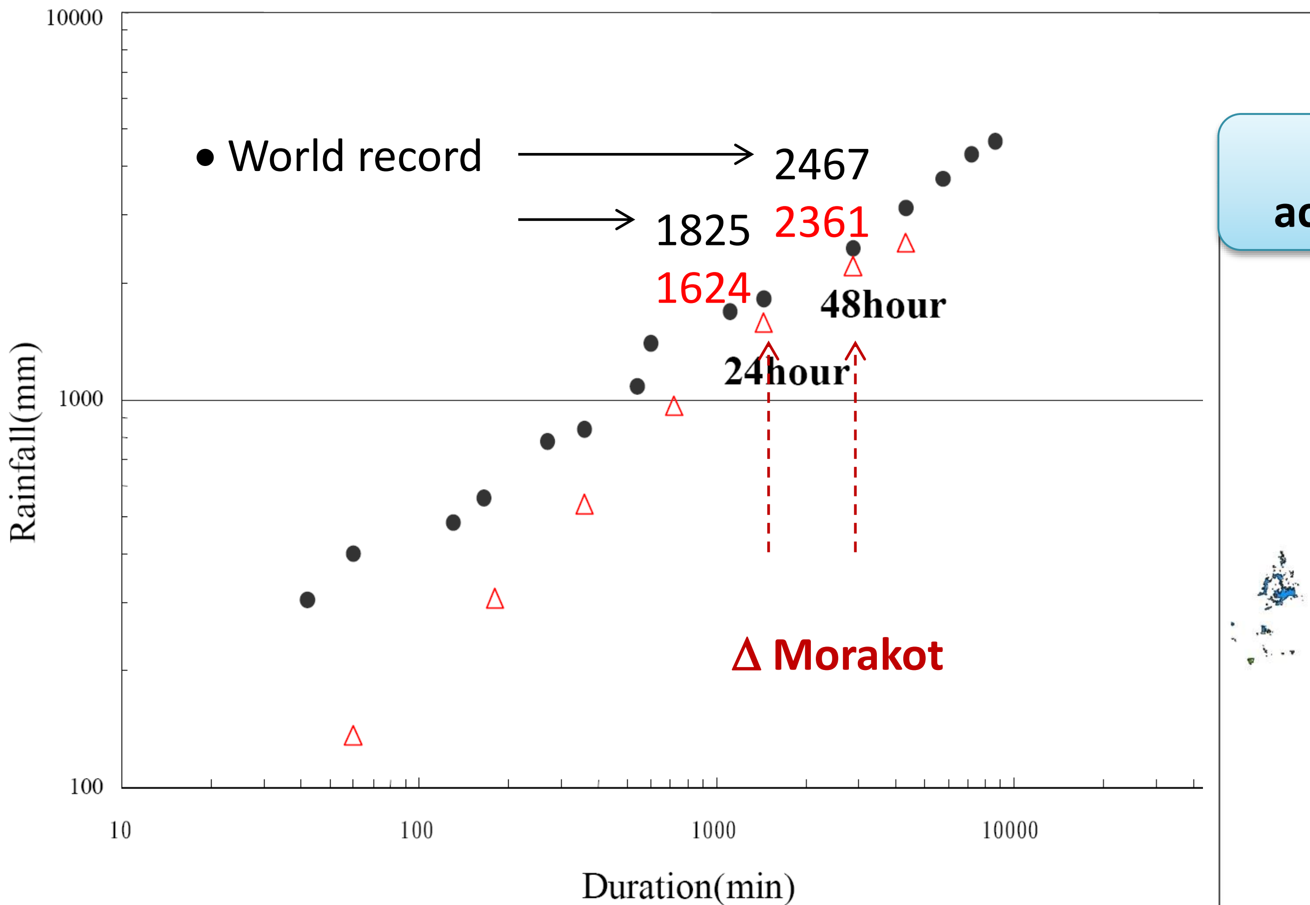
1994-year										
Duration(hrs)*	1	2	3	4	6	12	18	24	48	72
Hosoliau	6/18	6/18	6/18	6/18	6/18	10/9	10/9	10/9	10/9	10/9
Wutuh	6/18	6/18	6/18	9/12	9/12	9/12	9/12	9/12	9/12	9/12

19840624	WYNNE魏恩	26	7	0.269231
19850626	-	28	6	0.214286
19860919	ABBY艾貝	28	11	0.392857
19870727	ALEX亞力士	28	11	0.392857
19880813	-	18	10	0.555556
19890912	SARAH莎拉	25	20	0.8
19900623	OFELIA歐菲莉	28	16	0.571429
19910622	-	26	8	0.307692
19920830	POLLY寶莉	27	18	0.666667
19930912	ABE亞伯	20	10	0.5
19940803	CAITLIN凱特琳	28	19	0.678571
19950608	DEANNA菽安娜	28	13	0.464286
19960731	HERB賀伯	28	21	0.75
19970828	AMBER安珀	26	13	0.5
19980804	OTTO奧托	27	14	0.518519
19990811	-	27	6	0.222222
20000822	BILIS碧利斯	27	17	0.62963
20010917	NARI納莉	27	13	0.481481
20020805	-	28	12	0.428571
20030606	-	28	13	0.464286
20040702	MINDULLE敏督利	28	24	0.857143
20050718	HAITANG海棠	28	17	0.607143
20060609	-	28	19	0.678571
20071006	KROSA柯羅莎	28	9	0.321429
20080717	KALMAEGI卡玫基	28	18	0.642857
20090808	MORAKOT莫拉克	27	26	0.962963
20100919	FANAPI凡那比	27	18	0.666667

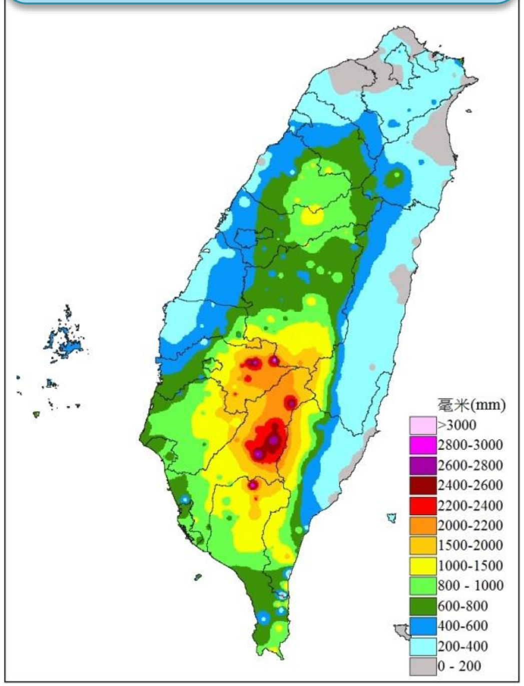
Proportion of rainfall stations observing annual max rainfalls.

- **A large number of rain gauges maintained by CWB and WRA. Some of them have record length longer than 50 years. Most of them have less than 30 years record length .**
- **Design rainfalls play a key role in studies related to climate change and disaster mitigation.**
- **Problems in rainfall frequency analysis**
  - Short record length (less than 30 years) (**small sample size**)
  - Record breaking rainfall extremes (**presence of extreme outliers**)
    - Typhoon Morakot

• Typhoon Morakot (2009)



0805-0810
accumulated rainfall



- Catastrophic storm rainfalls (or extraordinary rainfalls) often are considered as extreme outliers. Whether or not such rainfalls should be included in site-specific frequency analysis is disputable.
  - 24-hr annual max. rainfalls (Morakot) of 2009
    - 甲仙 1077 mm
    - 泰武 1747 mm
    - 大湖山 1329 mm
    - 阿禮 1237 mm

- Frequency analysis of 24-hr **annual maximum rainfalls (AMR)** at Jia-Sien station using **50 years** of historical data
  - 1040mm/24 hours (by Morakot) excluding Morakot – **901 years** return period
  - Return period inclusive of Morakot – **171 years**

The same amount (1040mm/24 hours) was found to be associated with a return period of **more than 2000 years** by another study which used 25 years of annual maximum rainfalls.

Morakot rainfalls were not included in the frequency analysis.

表 4.1 莫拉克颱風高屏溪各雨量站不同降雨延時降雨量頻率分析結果表

流域	雨量站	鄉鎮名稱	24小時		48小時		72小時			
			Rainfall (mm)	Return period	Rainfall (mm)	Return period	Rainfall (mm)	Return period	Total Rainfall	Record length
高屏溪	屏東(5)	屏東縣屏東市	667.0	141	886.0	124	947.0	159	959.0	38
	美濃(2)	高雄縣美濃鎮	507.0	>2000	749.0	>2000	828.0	>2000	871.0	19
	屏東	屏東縣屏東市	666.0	140	906.0	143	974.5	197	990.0	38
	溪埔	高雄縣大樹鄉	729.5	271	994.5	265	1057.5	378	1076.5	38
	旗山	高雄縣旗山鎮	621.0	>2000	813.0	>2000	854.5	>2000	881.0	15
	尾寮山	屏東縣三地門	1414.5	>2000	2215.5	>2000	2564.0	>2000	2701.0	21
	甲仙	高雄縣甲仙鄉	1077.5	>2000	1601.0	>2000	1856.0	>2000	1916.0	25
	古夏	屏東縣三地門鄉	683.5	>2000	946.0	>2000	1061.5	>2000	1127.0	25
	美濃	高雄縣美濃鎮	633.5	>2000	878	>2000	955.5	>2000	989.5	15
	里港	屏東縣里港鄉	710.5	>2000	955.5	>2000	1018	>2000	1039.5	15
	上德文	屏東縣三地門鄉	1185.5	>2000	1968.0	>2000	2194.5	>2000	2255.0	25
	新圍	屏東縣鹽埔鄉	578.0	148	757.5	>2000	806.5	565	830.5	25
	月眉	高雄縣杉林鄉	744.0	>2000	1081.0	>2000	1205.0	>2000	1246.5	19
	吉東	高雄縣美濃鎮	547.5	>2000	728.0	>2000	789.0	>2000	820.5	19
	大津	高雄縣六龜鄉	738.5	>2000	1072.0	>2000	1241.0	>2000	1314.0	21

資料來源：「莫拉克颱風暴雨量及洪流量分析」，經濟部水利署，民國98年9月。原資料尚有濁水溪、北港溪、朴子溪、八掌溪、急水溪、曾文溪、鹽水溪、二仁溪、東港溪、四重溪、林邊溪、知本溪等各雨量站不同降雨延時降雨量頻率分析結果。

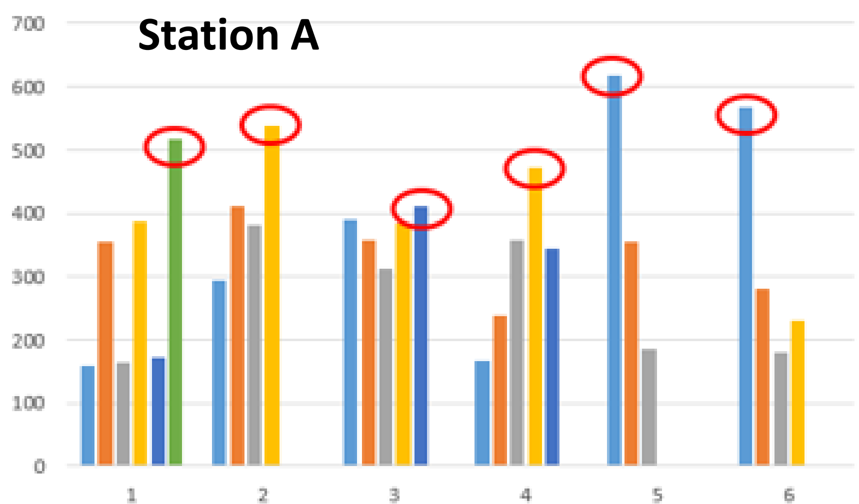
- **Concurrent occurrences of extraordinary rainfalls at different rain gauges**
  - Several stations had 24-hr rainfalls exceeding 100-yr return period.
  - Site-specific events of 100-yr return period.
  - What is the return period of the event of multi-site 100-yr return period?
    - (100)^4 = 100,000,000 years (4 sites), assuming independence
  - Redefining extreme events
    - multi-site extreme events w.r.t. specified durations and rainfall thresholds
    - Spatial covariation of rainfall extremes

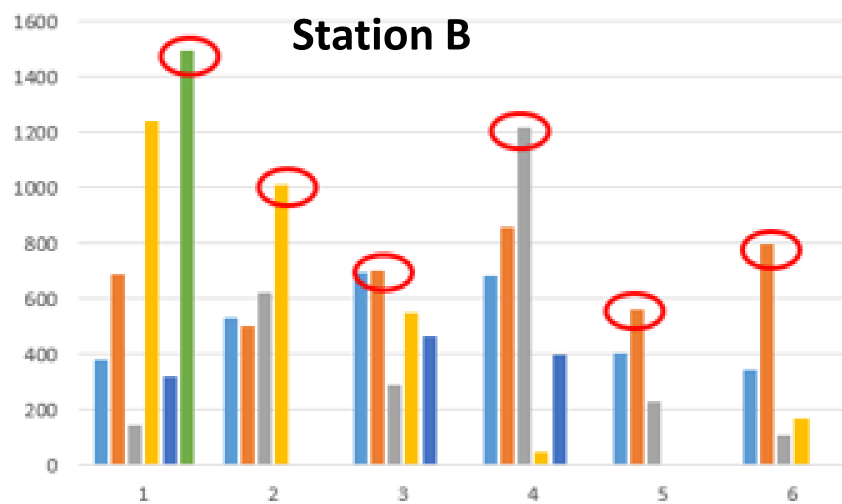
- **Spatial covariation of rainfall extremes**
  - By using site-specific annual maximum rainfall series for frequency analysis implies a significant loss of valuable information.

24-hr Event Maximum Rainfalls

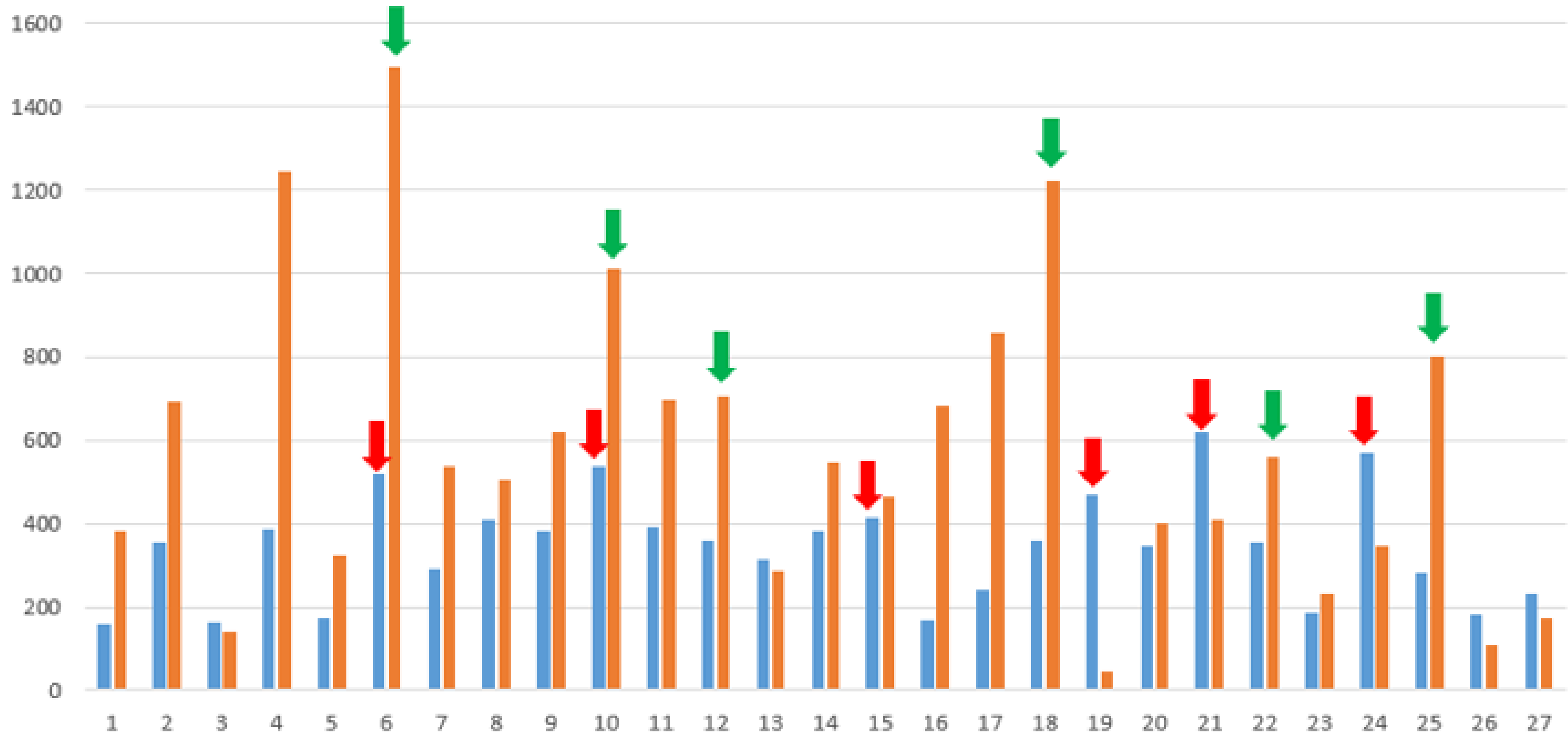
	Station A						Station B					
Year	Event-1	Event-2	Event-3	Event-4	Event-5	Event-6	Event-1	Event-2	Event-3	Event-4	Event-5	Event-6
1	160	355	164	387	173	518	383	693	142	1242	321	1493
2	293	410	383	537			535	505	621	1009		
3	390	358	313	384	412		696	704	289	548	463	
4	168	239	359	471	344		683	858	1219	47	400	
5	617	356	186				408	560	230			
6	567	280	180	231			347	800	111	171		

Probability distribution of Annual Maximum Rainfalls (Small sample size)





Probability distribution of Event-Maximum Rainfalls (Large sample size)



- **Coping with outliers**
  - Regional Frequency Analysis
  - Stochastic simulation of multi-site event-maximum rainfalls

Fundamental concept of regional frequency analysis

- **For areas with short record length or without rainfall or flow measurements, hydrological frequency analysis needs to be conducted using data from sites of similar hydrological characteristics.**
- **Data observed at different sites within a "*homogeneous region*" can be combined and used in *regional* frequency analysis.**

General procedures of regional frequency analysis (Hosking and Wallis, 1997)

1. **Data screening**
   - Correctness check
   - Data should be stationary over time.
2. **Identifying homogeneous regions**
   - A set of characteristic variables should be chosen and used for delineation of homogeneous regions.
   - Characteristic variables may include geographic and hydrological variables.
   - Homogeneous regions are often determined by cluster analysis.

3. Choice of an appropriate regional frequency distribution

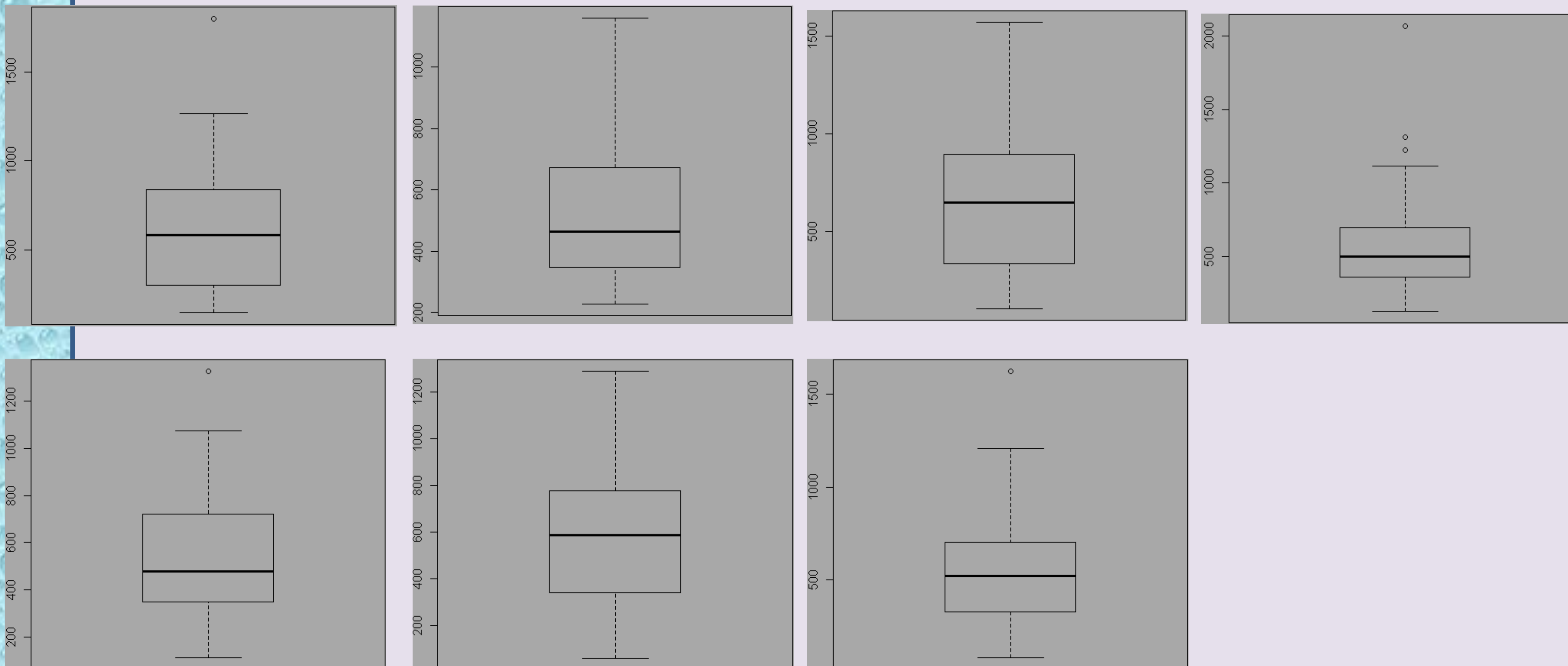
– GOF test using rescaled samples from different sites within the same homogeneous region.
– The chosen distribution not only should fit the data well but also yield quantile estimates that are robust to physically plausible deviations of the true frequency distribution from the chosen frequency distribution.

4. **Parameter estimation of the regional frequency distribution**
   - Estimating parameters of the site-specific frequency distribution
   - Estimating parameters of the regional frequency distribution using record-length weighted average.
5. **Calculating various quantiles of the hydrological variable under investigation.**

An experimental stochastic simulation

- We generated 7 random samples (sample size n = 40) of a gamma distribution with mean=600 and std dev =346.
- The data set is considered equivalent to annual max rainfalls (record length = 40) at 7 stations.
- Outliers were detected in four of the seven series.
- Return period of the maximum value of each individual AMR series was calculated.

Max value in AMR	1797.758	1159.976	1567.84	2066.767	1326.539	1288.869	1624.943



Max values	1797.758	1159.976	1567.84	2066.767	1326.539	1288.869	1624.943
RT (site-specific)	164.1526	59.64806	35.99707	262.0977	54.71051	26.05501	102.0624
RT (Population)	159.0214	13.98271	64.15083	475.1555	25.63451	22.3025	80.15404
RT (Regional freq)	180.9445	13.90993	69.13296	579.7057	26.26874	22.69228	87.5076

Study area and rainfall stations

25 rainfall stations in southern Taiwan. (1951 – 2010)
Not all stations have the same record length.
Annual maximum rainfalls (AMR) of various durations (1, 2, 6, 12, 18, 24, 48, 72 hours)

Event-maximum rainfalls of various design durations

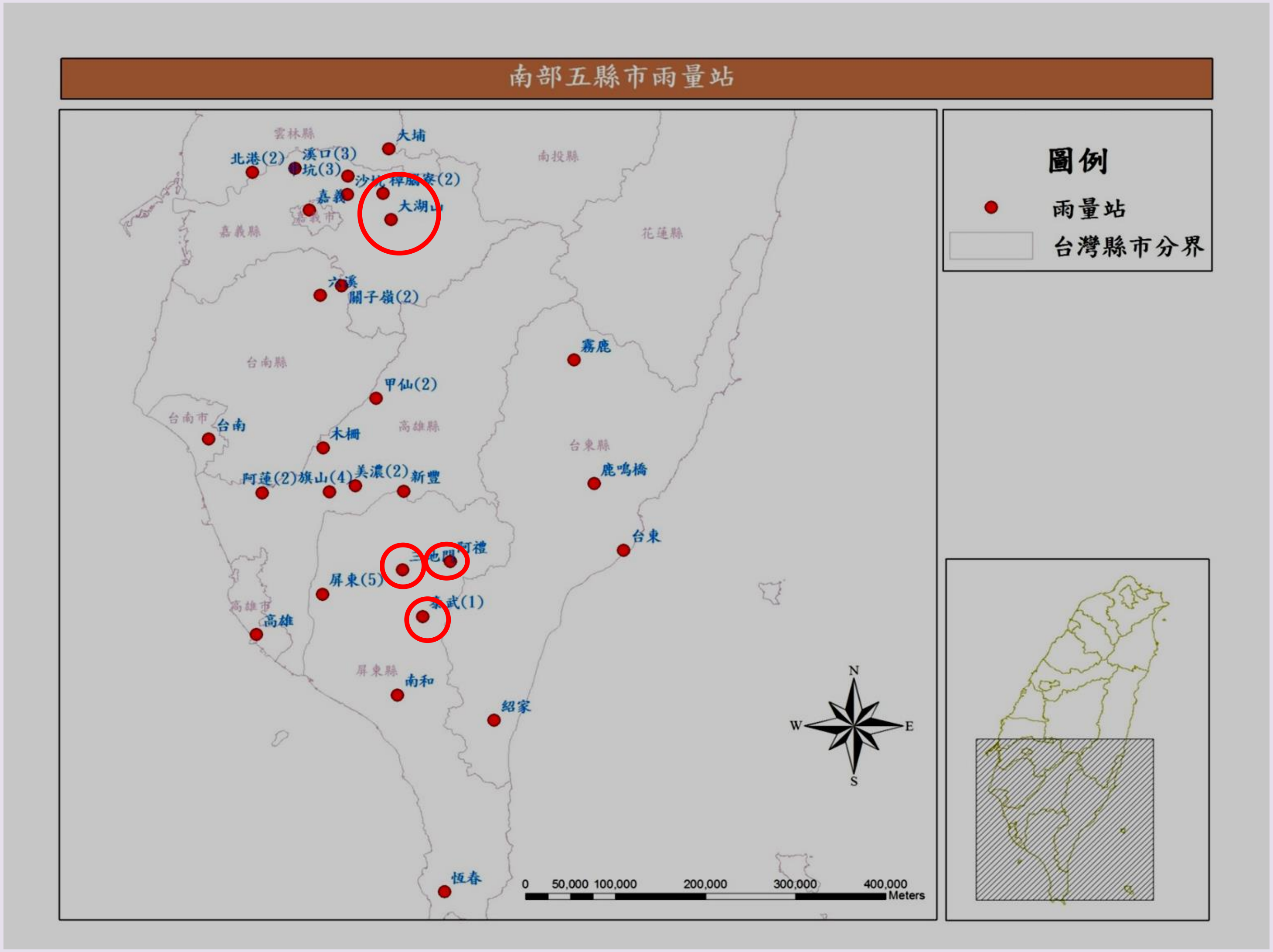
- Event-max 1, 2, 6, 12, 18, 24, 48, 72-hr typhoon rainfalls at individual sites.
- Approximately 120 events
- Complete series

Delineation of homogeneous regions (K-mean cluster analysis)

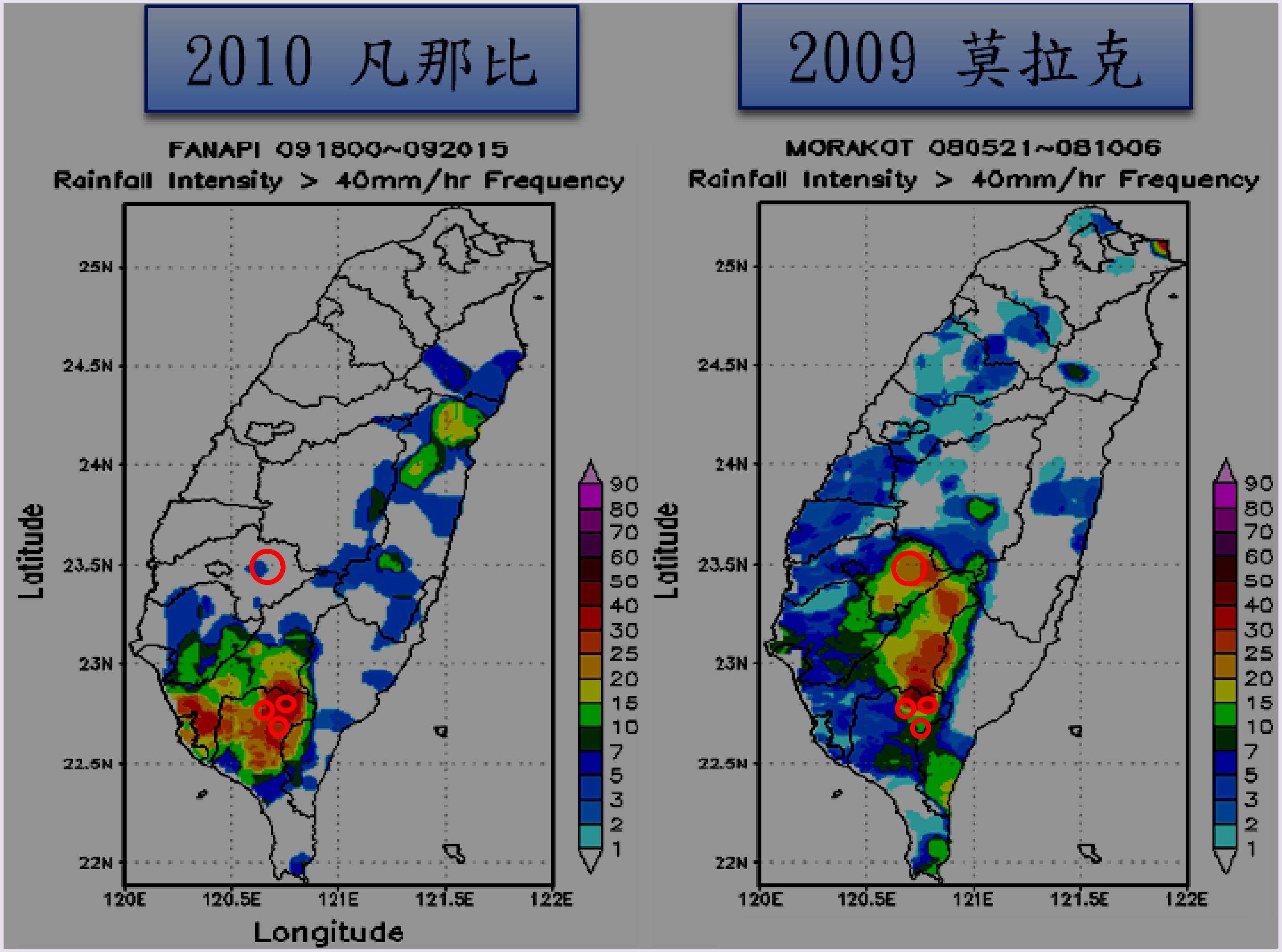
- **24 classification features (8 design durations x 3 parameters – mean, std dev, skewness)**
  - Normalization of individual features
- **Two homogeneous regions (25 stations)**

Regional frequency analysis

- **Delineating homogeneous regions**

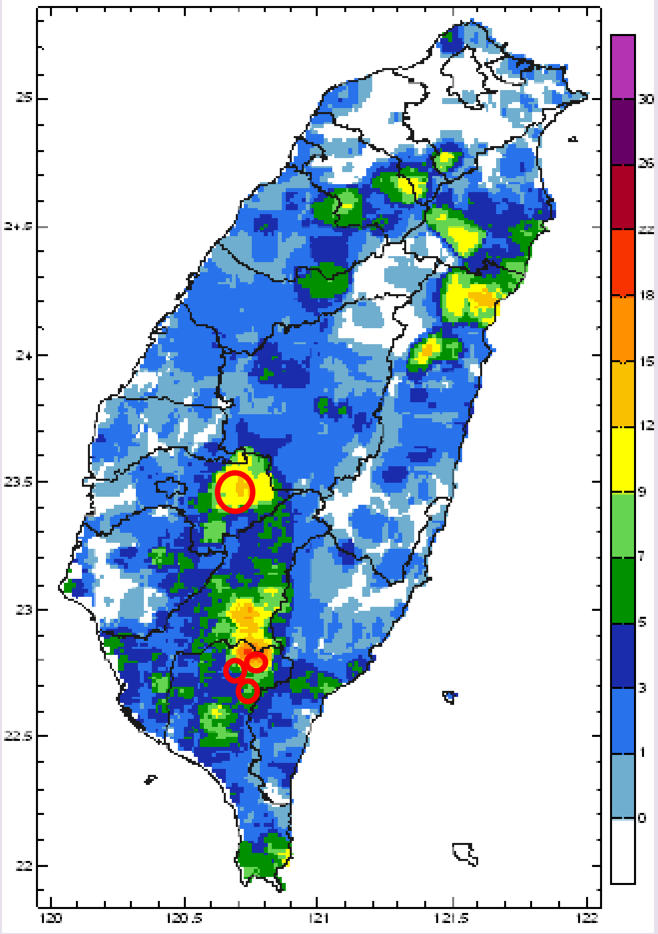


颱風強降雨頻率分布



2010 凡那比
2009 莫拉克
FANAPI 091800~092015
Rainfall Intensity > 40mm/hr Frequency
MORAKOT 080521~081006
Rainfall Intensity > 40mm/hr Frequency
Latitude
Longitude
25N
24.5N
24N
23.5N
23N
22.5N
22N
120E
120.5E
121E
121.5E
122E
90
80
70
60
50
40
30
25
20
15
10
7
5
3
2
1

Hot spots for occurrences of extreme rainfalls

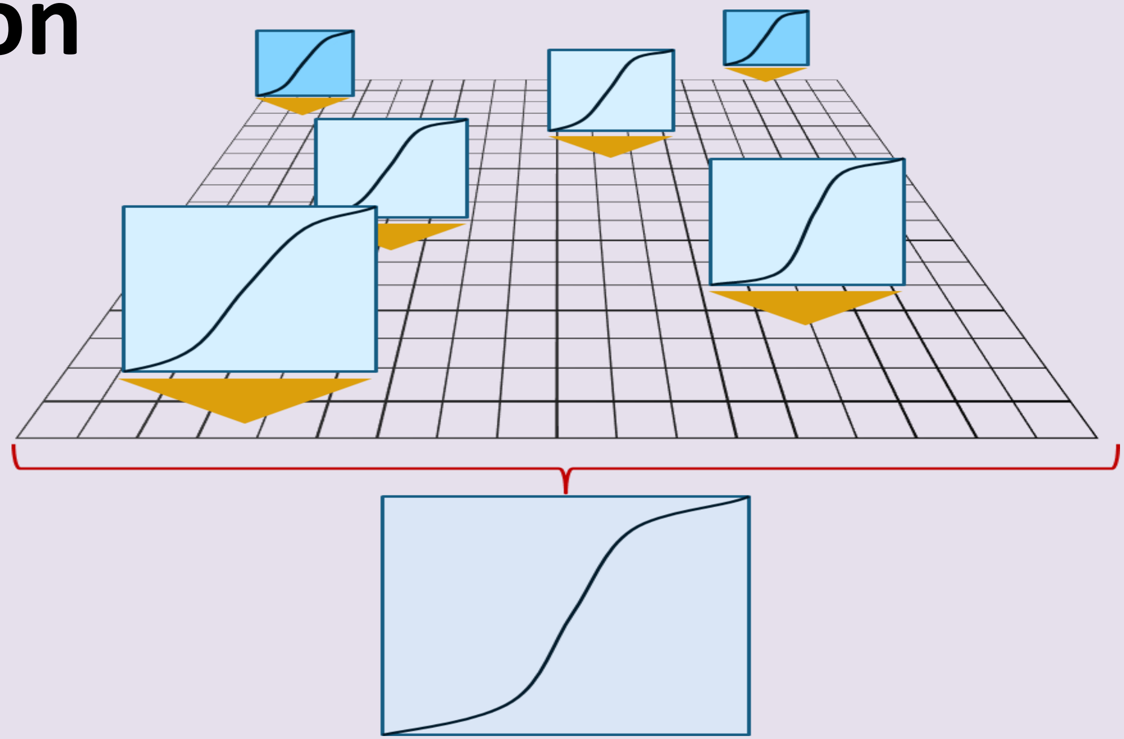


1992 – 2010
Number of extreme typhoon events

Rescaled variables for regional frequency analysis – Frequency factors

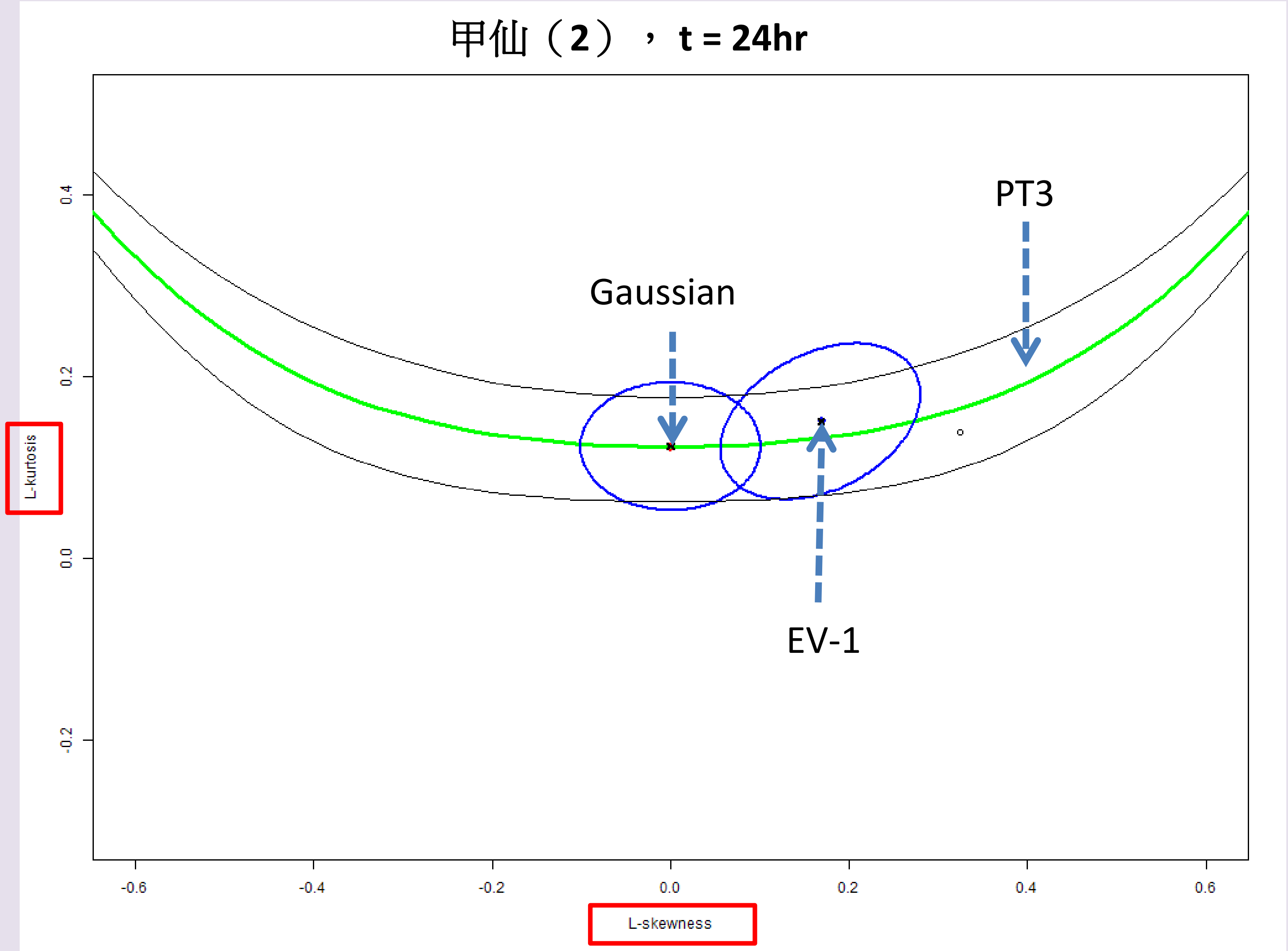
$$K_{ijk} = \frac{x_{ijk} - \mu_{ik}}{\sigma_{ik}}$$

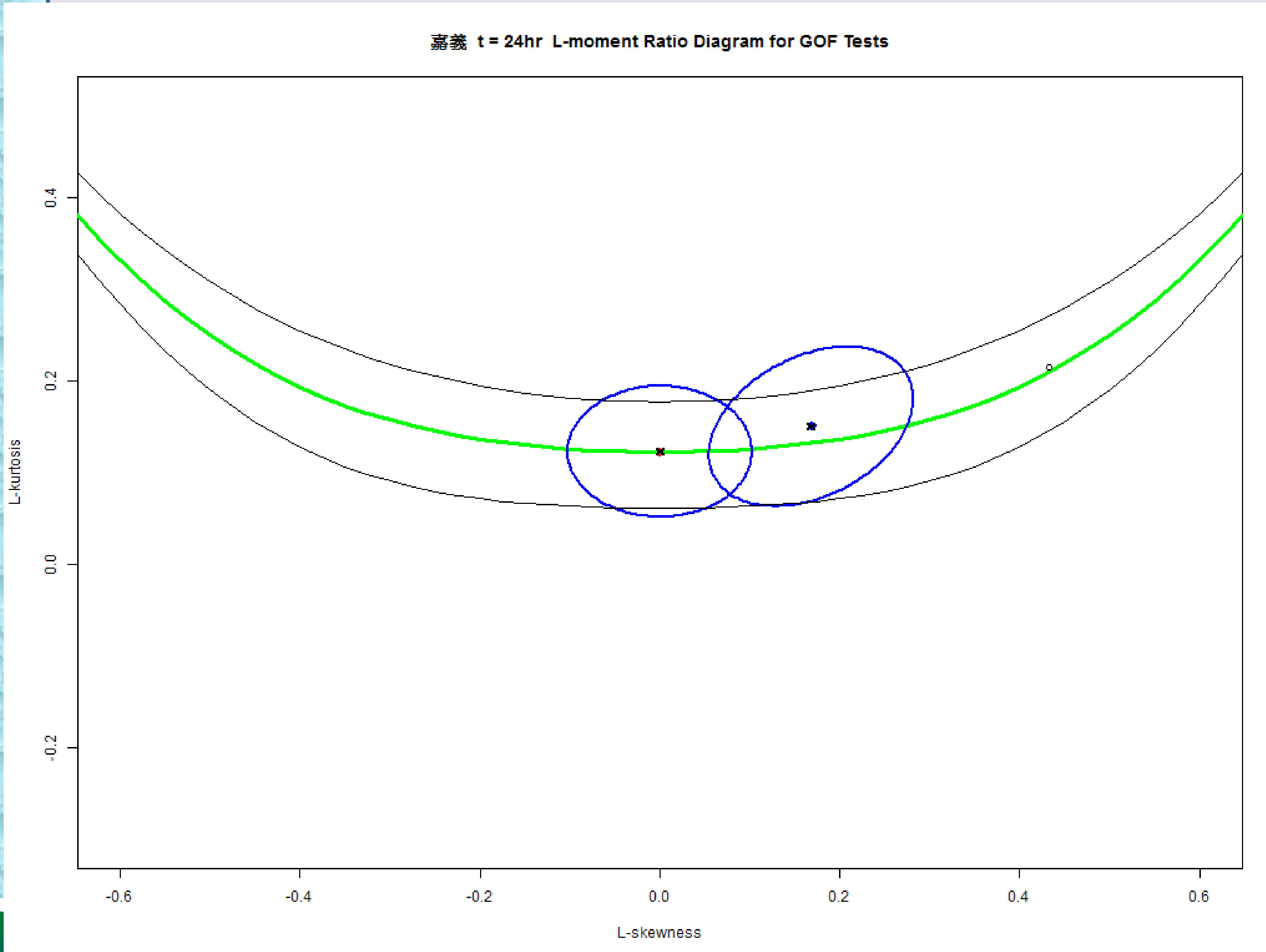
Goodness-of-fit test and parameters estimation



- L-moment ratio diagrams (LMRD) for goodness-of-fit test
  - Pearson Type 3 distribution
- Parameters estimation
  - Method of L-moments
  - Record-length-weighted regional parameters

L-moment-ratio diagram GOF test




嘉義 t = 24hr L-moment Ratio Diagram for GOF Tests
L-kurtosis
L-skewness
0.4
0.2
0.0
-0.2
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6

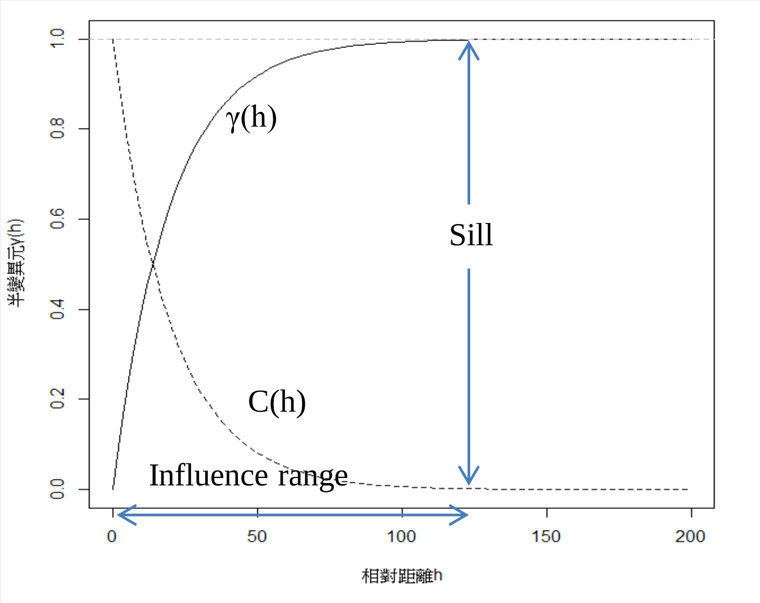
Covariance structure of the random field

- Covariance matrices are semi-positive definite.
- Experimental covariance matrices often do not satisfy the semi-positive definite condition.
- Modeling the covariance structure of frequency factors (event-max rainfalls) by variogram modeling.

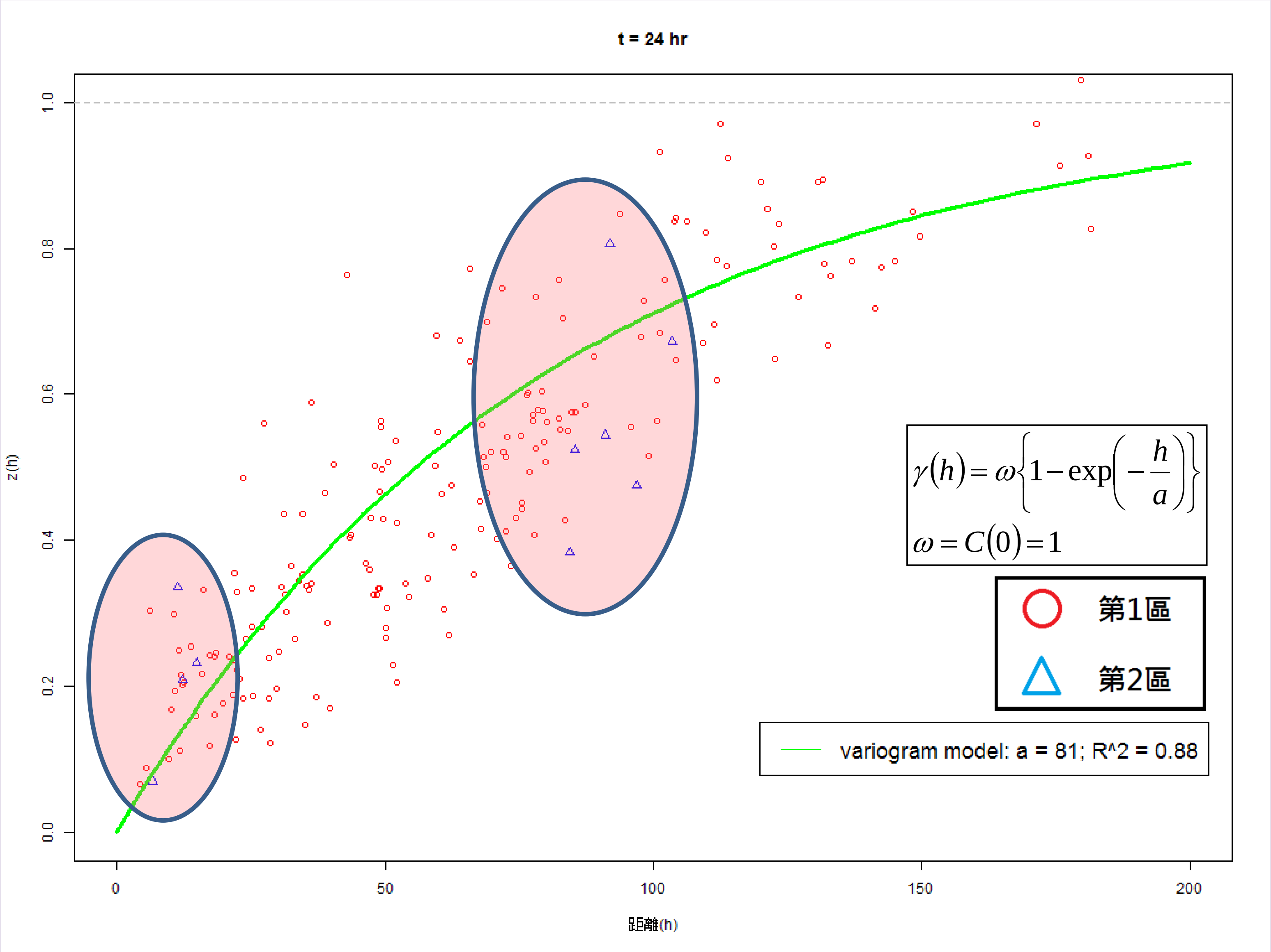
Relationship between semivariogram and covariance function

$$\gamma(h)=\frac{1}{2}Var[Z(x+h)-Z(x)]$$
$$=C(0)-C(h)$$

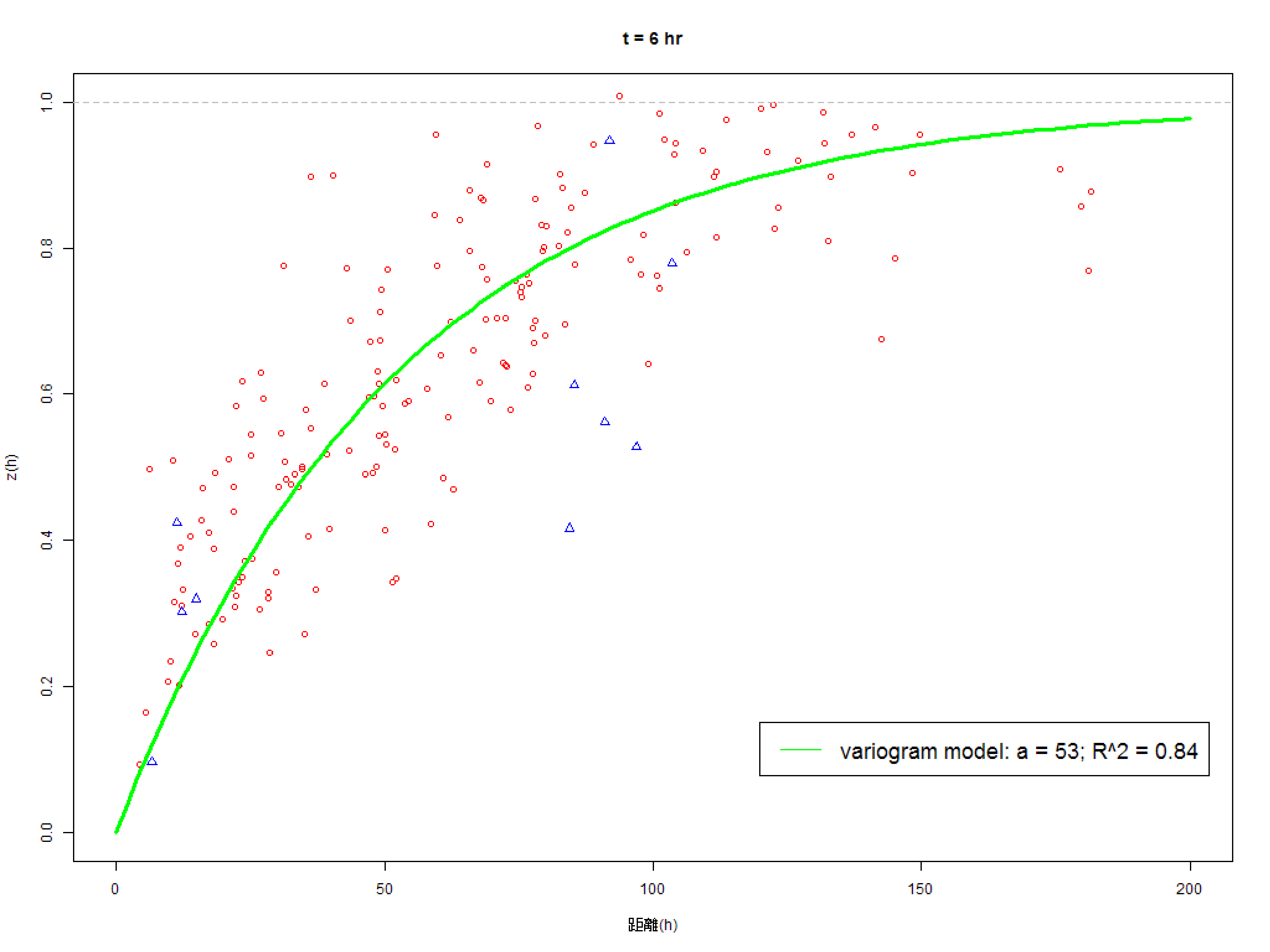
$$C(x_i,x_j)=C(h)$$
$$sill=C(0)=1$$



Semi-variogram modeling (24-hr EMR)



Semi-variograms of EMRs of other durations



Stochastic Simulation of Multi-site Event-Max Rainfalls

- **Pearson type III (Non-Gaussian) random field simulation**
- **Covariance Transformation Approach**
  - Covariance matrix of multivariate PT3 distribution
  - Covariance matrix of multivariate standard Gaussian distribution
  - Multivariate Gaussian simulation
  - Transforming simulated multivariate Gaussian realizations to PT3 realizations

$\rho_{XY} \sim \rho_{UV}$ Conversion

$$\rho_{XY} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$
$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3$$

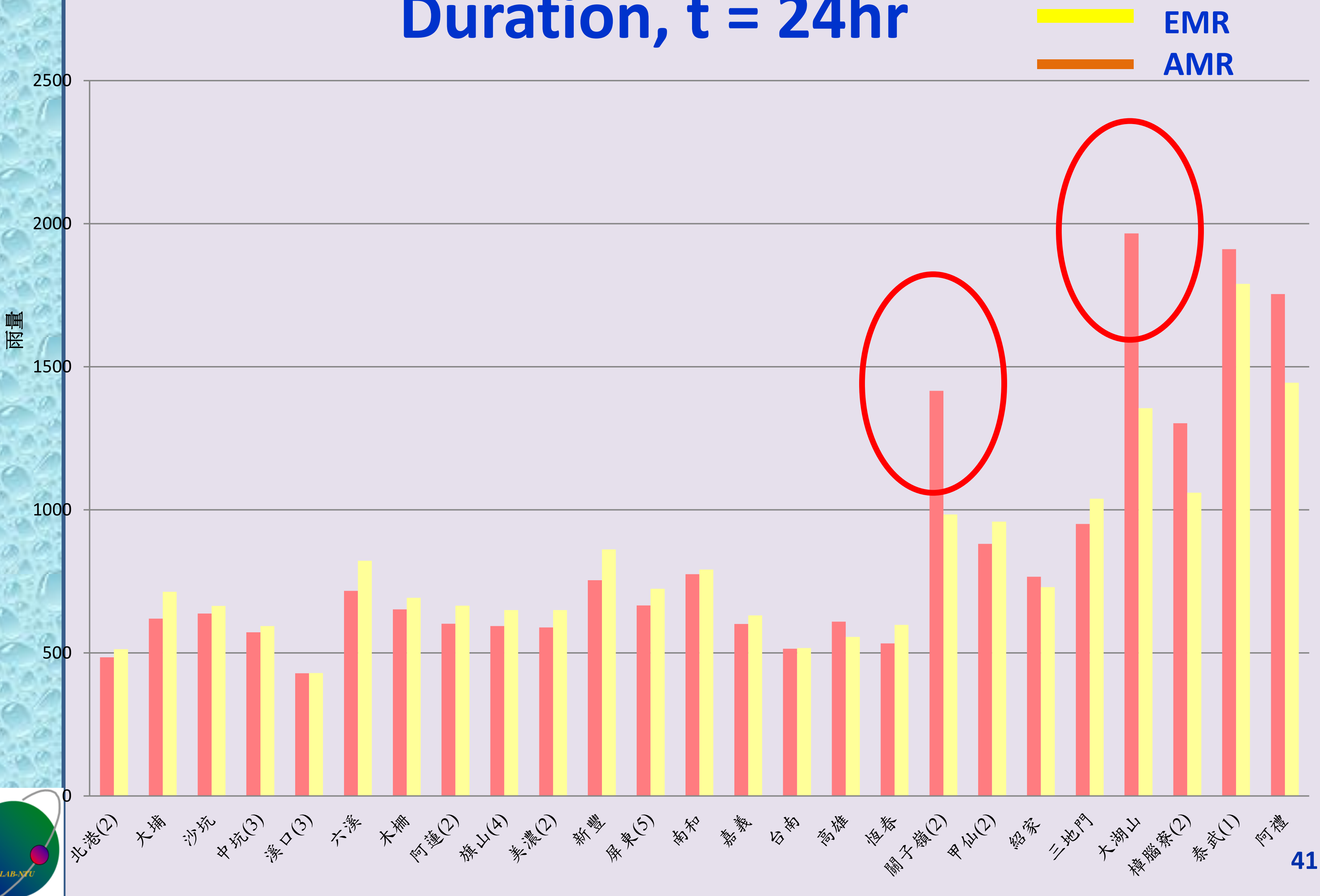
$$A_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 \quad B_X = \frac{\gamma_X}{6} - \left(\frac{\gamma_X}{6}\right)^3 \quad C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2$$

$$A_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 \quad B_Y = \frac{\gamma_Y}{6} - \left(\frac{\gamma_Y}{6}\right)^3 \quad C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2$$

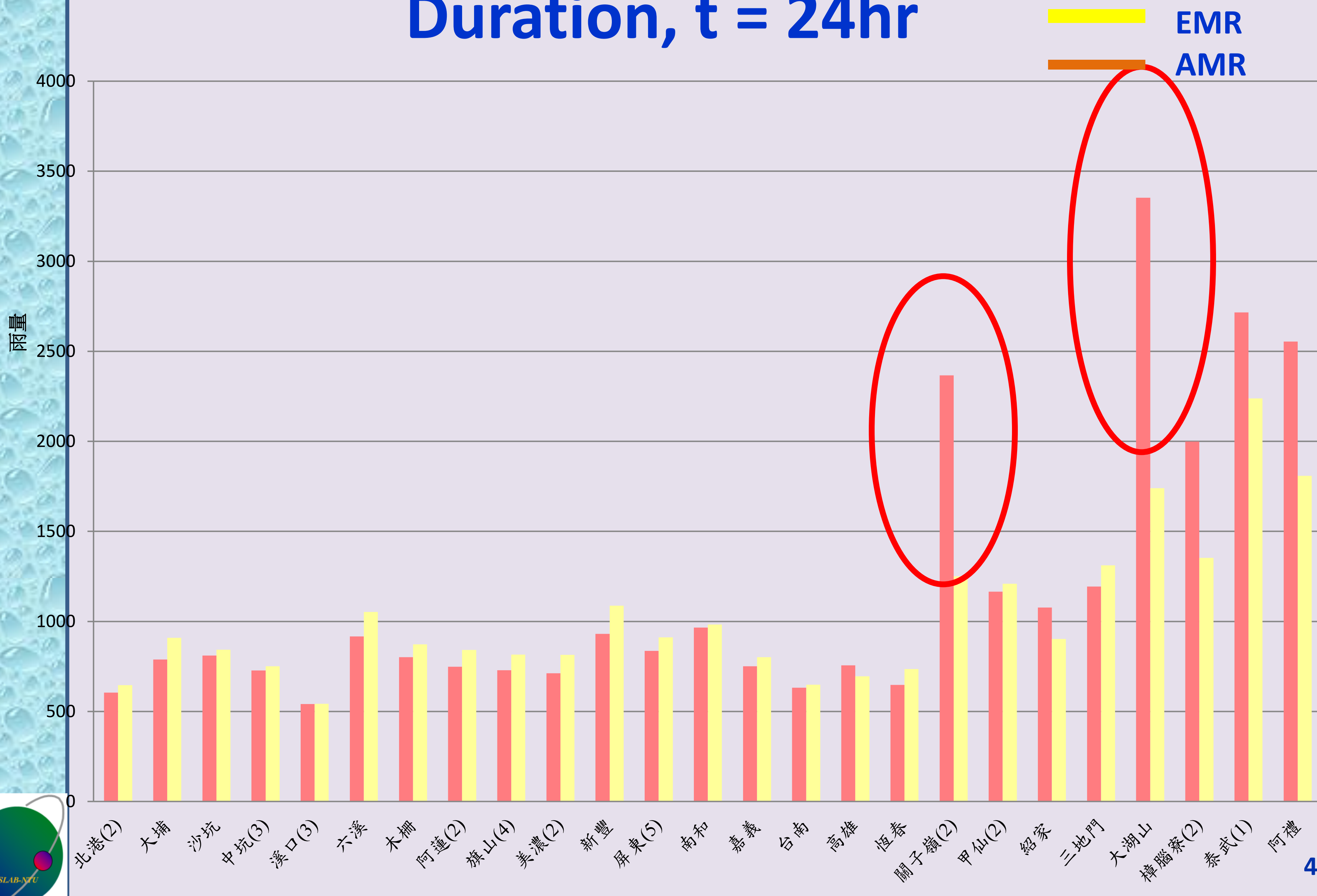
- **Each simulation run generated one sample of t-hr multi-site event maximum rainfalls.**
- **Simulated samples preserved the spatial covariation of multi-site EMRs as well as the marginal distributions.**
- **10,000 samples were generated.**
  - Multi-site t-hr EMRs of 10,000 typhoon events.
- **The number of typhoons vary from one year to another.**
  - **Annual count of typhoons is a random variable.**

- **Determination of t-hr rainfall of T-yr return period**
  - Average number of typhoons per year, m = 2.43
  - Return period, T=100 years
  - Exceedance probability of the event-max rainfalls, p_E=1/(100*2.43)

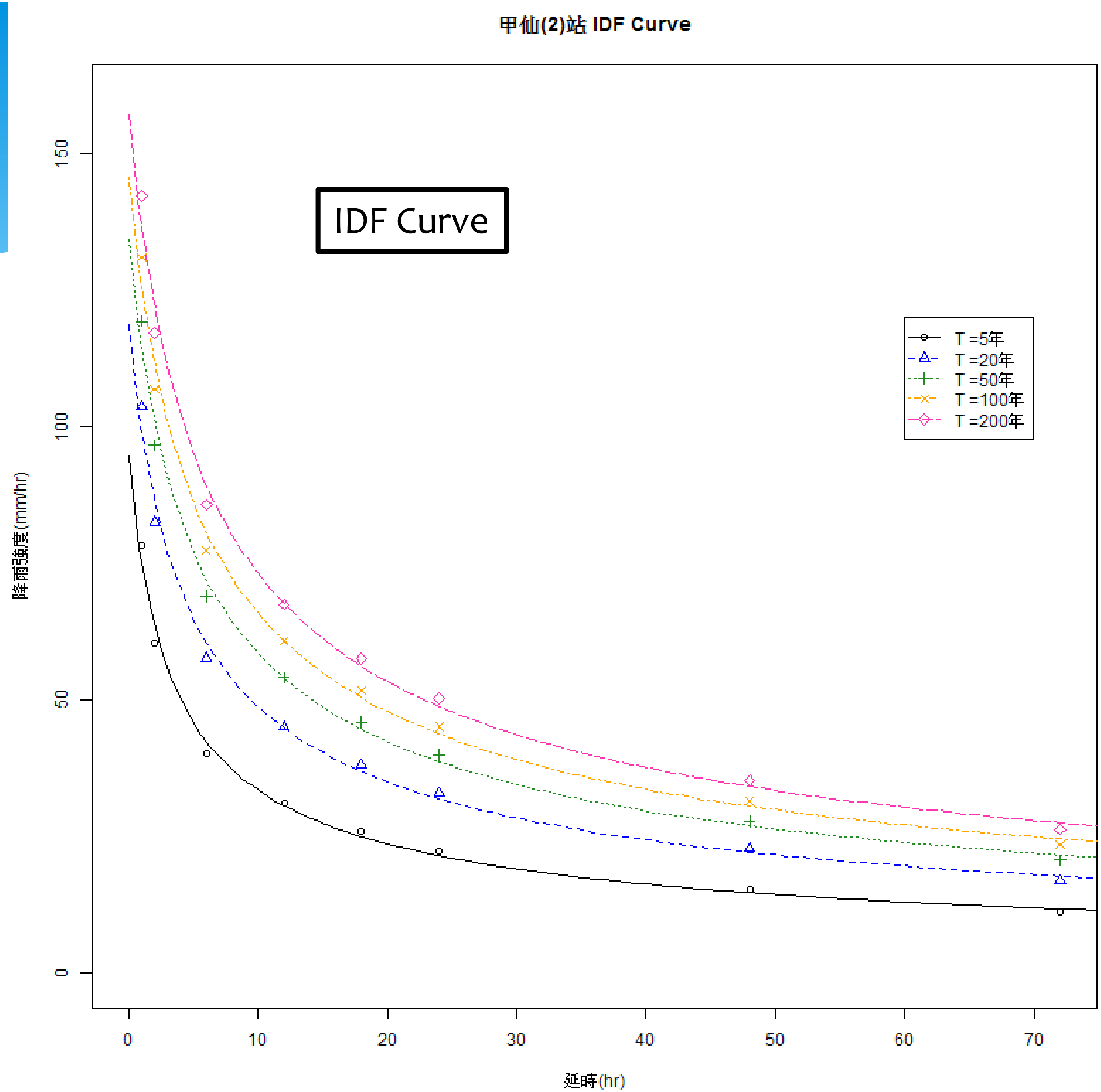
Return period, T = 50yr
Duration, t = 24hr



Return Period, T = 200yr
Duration, t = 24hr



	24hr雨量(mm)	24hr重現期(年)	48hr雨量(mm)	48hr重現期(年)	72hr雨量(mm)	72hr重現期(年)
北港(2)	358	11	433	11	460	12
大埔	720	53	921	57	1069	118
沙坑	607	32	759	31	888	52
中坑(3)	497	24	636	26	744	42
溪口(3)	324	15	403	16	450	24
六溪	791	43	1024	55	1122	59
木柵	778	102	1037	54	1188	80
阿蓮(2)	630	41	868	42	924	41
旗山(4)	519	17	760	23	819	22
美濃(2)	338	4	509	6	569	6
新豐	908	63	1179	51	1195	36
屏東(5)	676	40	878	45	938	45
南和	614	15	988	39	1100	62
嘉義	526	22	646	17	697	24
台南	535	61	709	53	736	42
高雄	538	40	755	57	801	53
恆春	528	27	715	34	730	28
關子嶺(2)	1098	92	1490	126	1683	221
甲仙(2)	1040	65	1614	177	1915	204
紹家	818	110	1176	141	1333	130
三地門	825	18	1109	25	1235	29
大湖山	1329	48	1958	89	2533	306
樟腦寮(2)	868	22	1395	55	1846	141
泰武(1)	1747	47	2938	121	3417	171
阿禮	1237	23	1908	38	2335	66

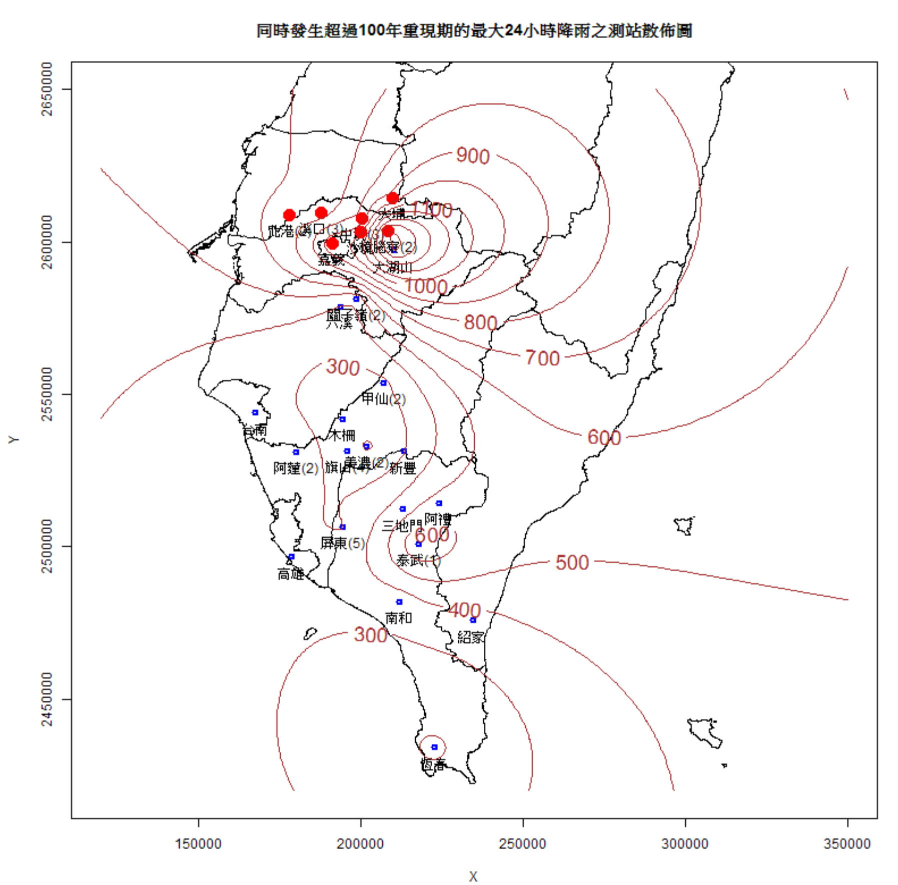

甲仙(2)站 IDF Curve
IDF Curve
T =5年
T =20年
T =50年
T =100年
T =200年
降雨強度(mm/hr)
延時(hr)
0
50
100
150
0
10
20
30
40
50
60
70

Estimating return period of multisite Morakot rainfall extremes

- **Four rain gauges within the Kaoping River Basin recorded 24-hr rainfalls close to 1,000mm (908, 1040, 825, 1237 mm).**
- **Define a multi-site extreme event**
  - over 1,000 mm 24-hr rainfalls at all four sites
  - Among the 10,000 simulated events, only 8 events satisfied the above requirement.
  - Average number of typhoons per year, m=2.43
  - Multi-site extreme event return period T = 514 years.

- **Further look at the simulation results**
  - Preserving the spatial pattern of rainfall extremes

Type A


同時發生超過100年重現期的最大24小時降雨之測站散佈圖
Y
2650000
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2500000
2450000
X
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250000
300000
350000
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1100
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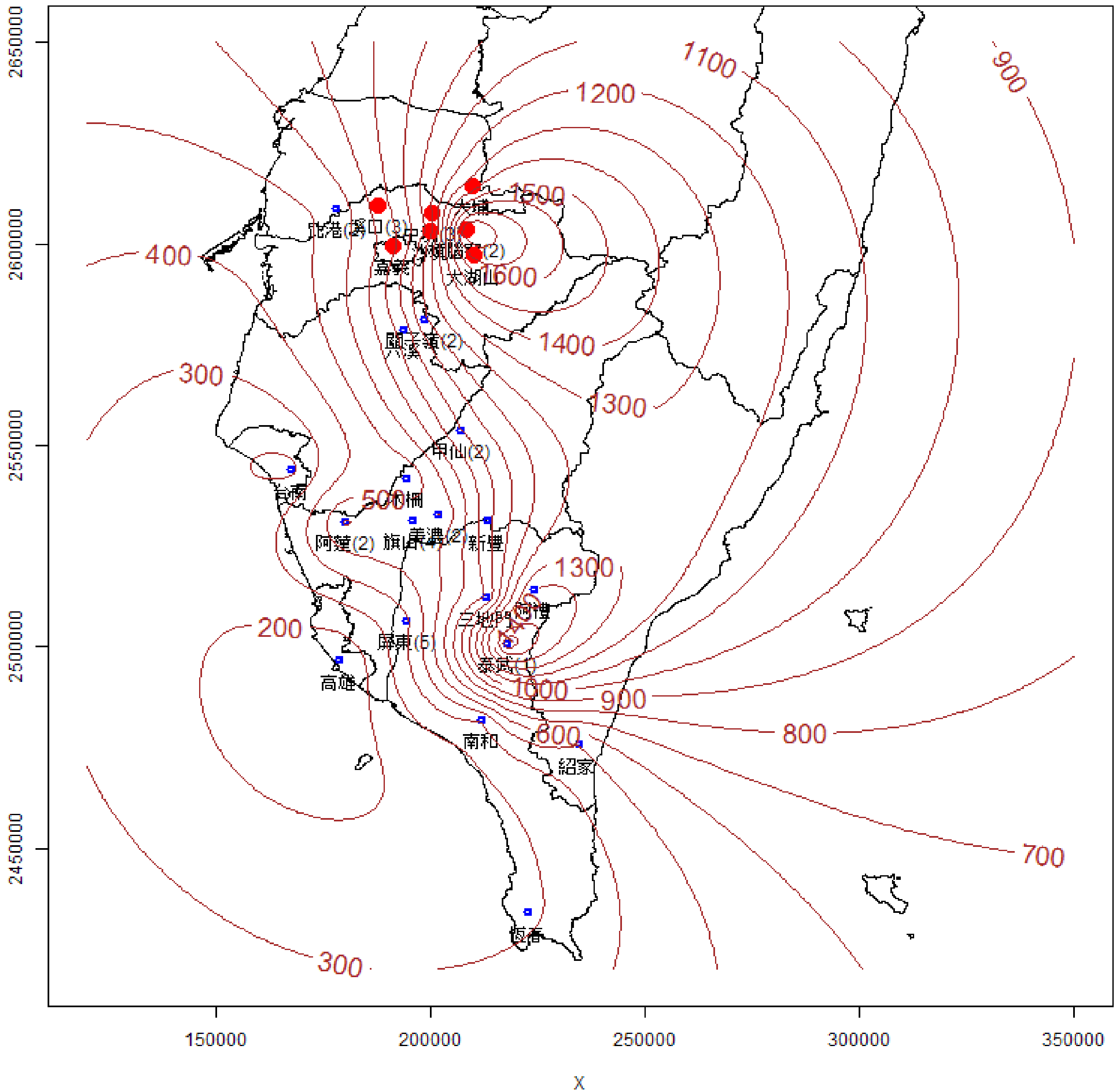
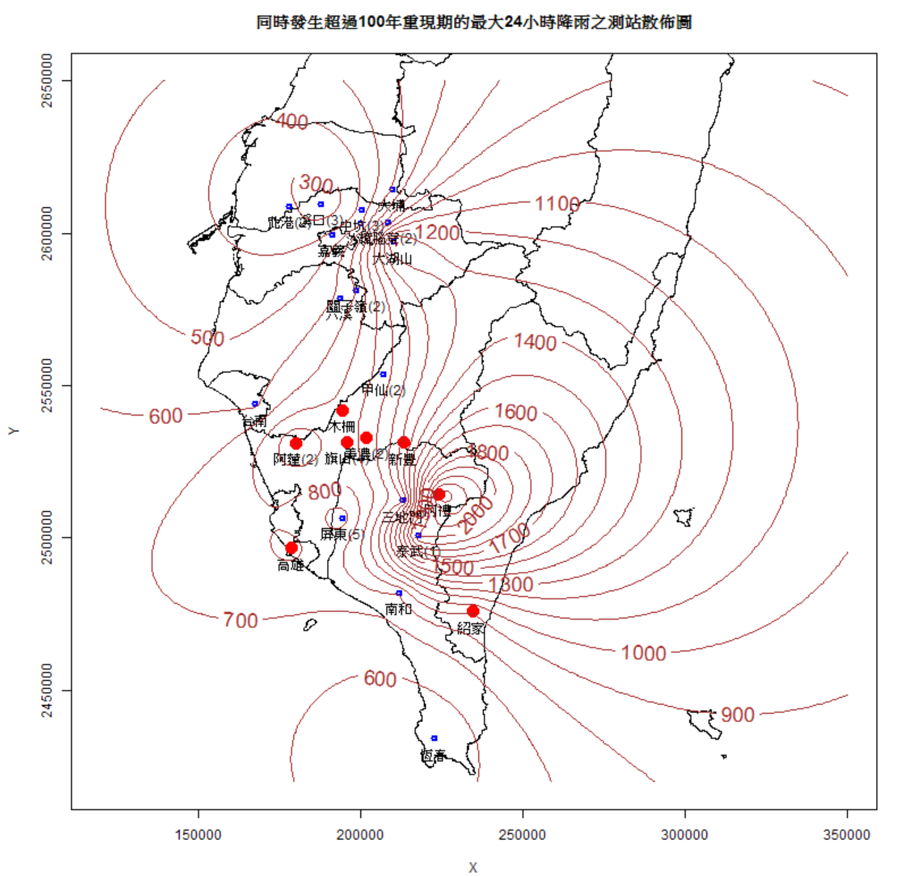
700
600
500
400
300
大湖山
甲仙(2)
木柵
台南
阿蓮(2)
新豐
三地門
阿禮
屏東(5)
高雄
南和
紹家
恆春

同時發生超過100年重現期的最大24小時降雨之測站散佈圖



Type B


同時發生超過100年重現期的最大24小時降雨之測站數佈圖
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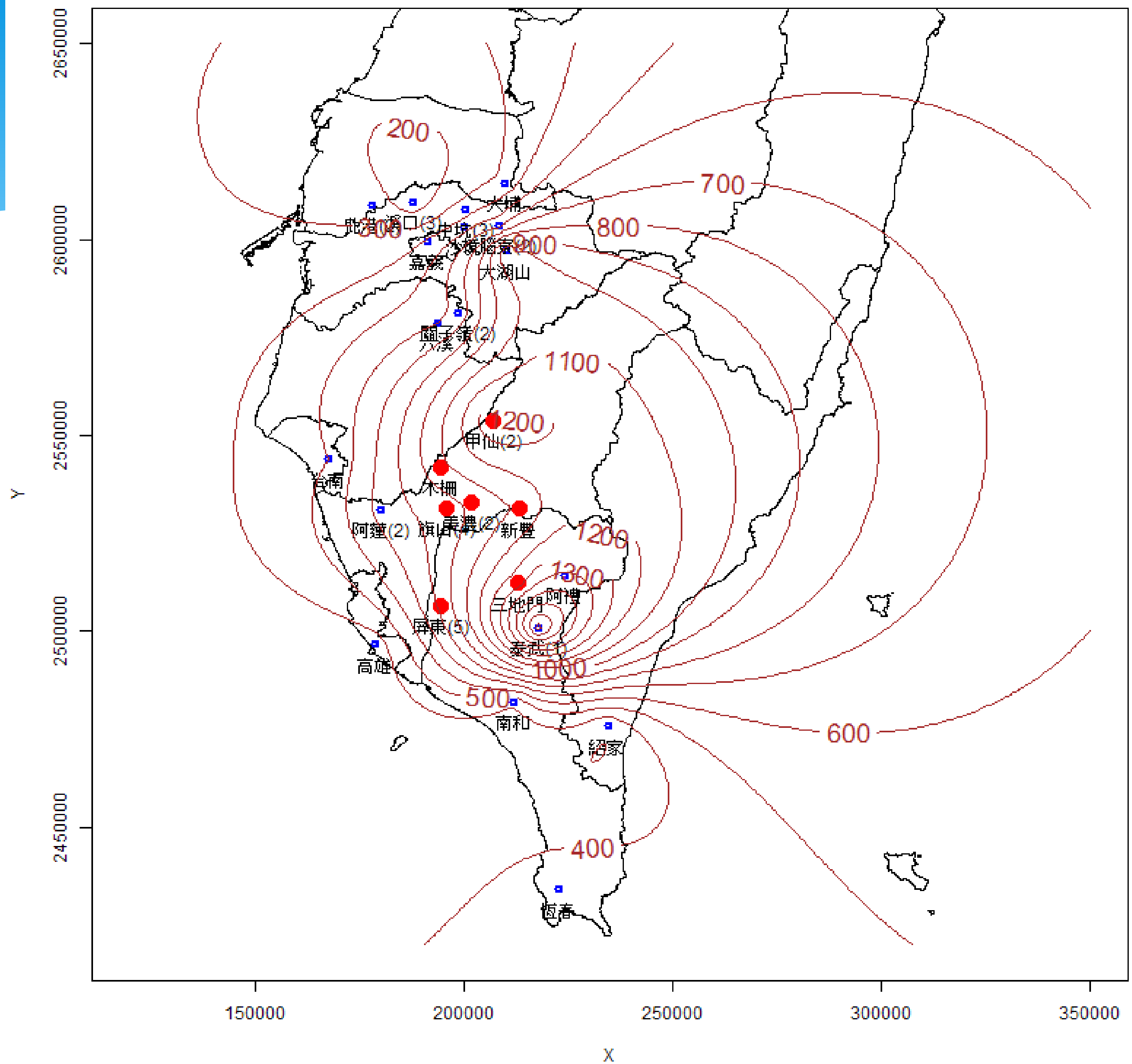
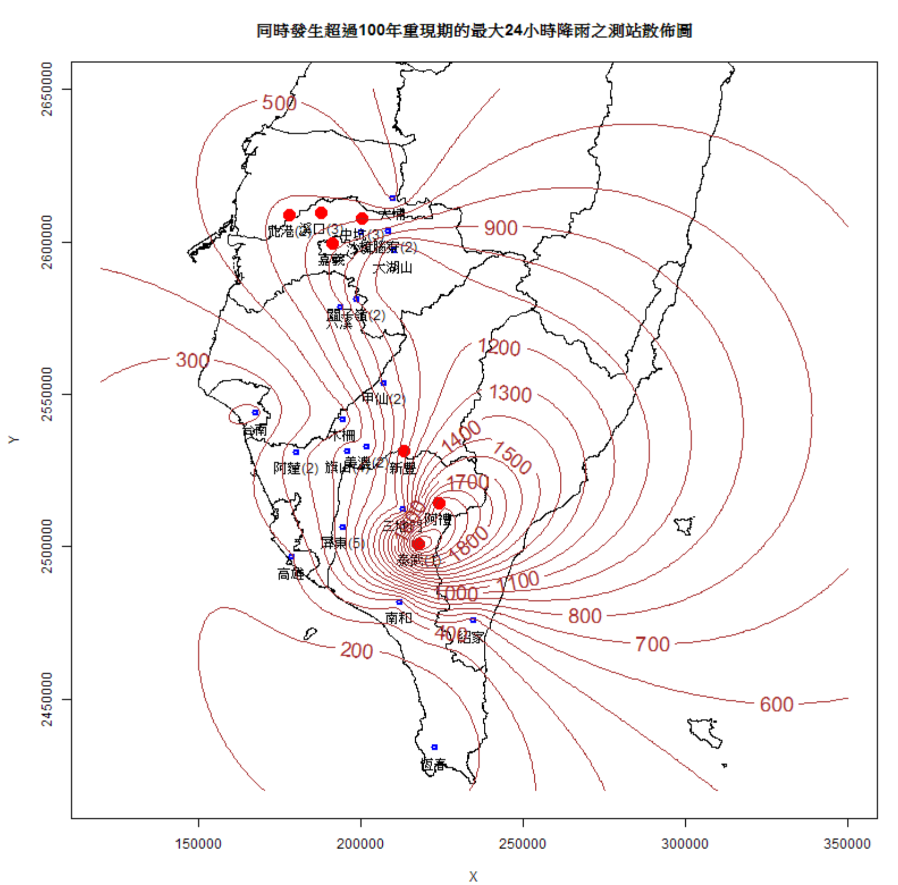
1600
1700
1800
2000
600
大埔
大湖山
甲仙(2)
台南
木柵
阿蓮(2)
新豐
屏東(5)
高雄
南和
紹家
泰武(1)
恆春

同時發生超過100年重現期的最大24小時降雨之測站散佈圖



Type C



同時發生超過100年重現期的最大24小時降雨之測站數佈圖
2650000
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1200
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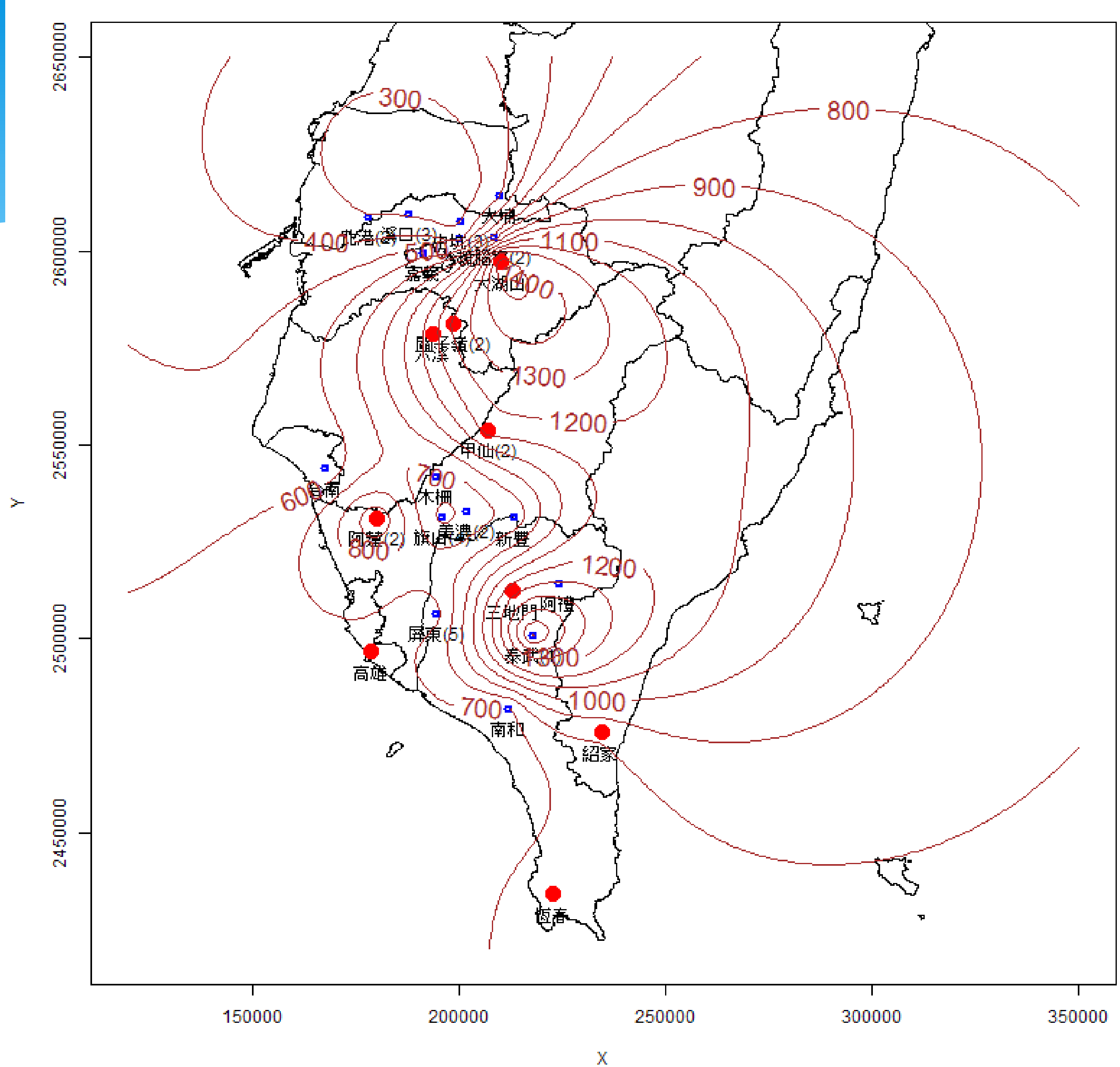
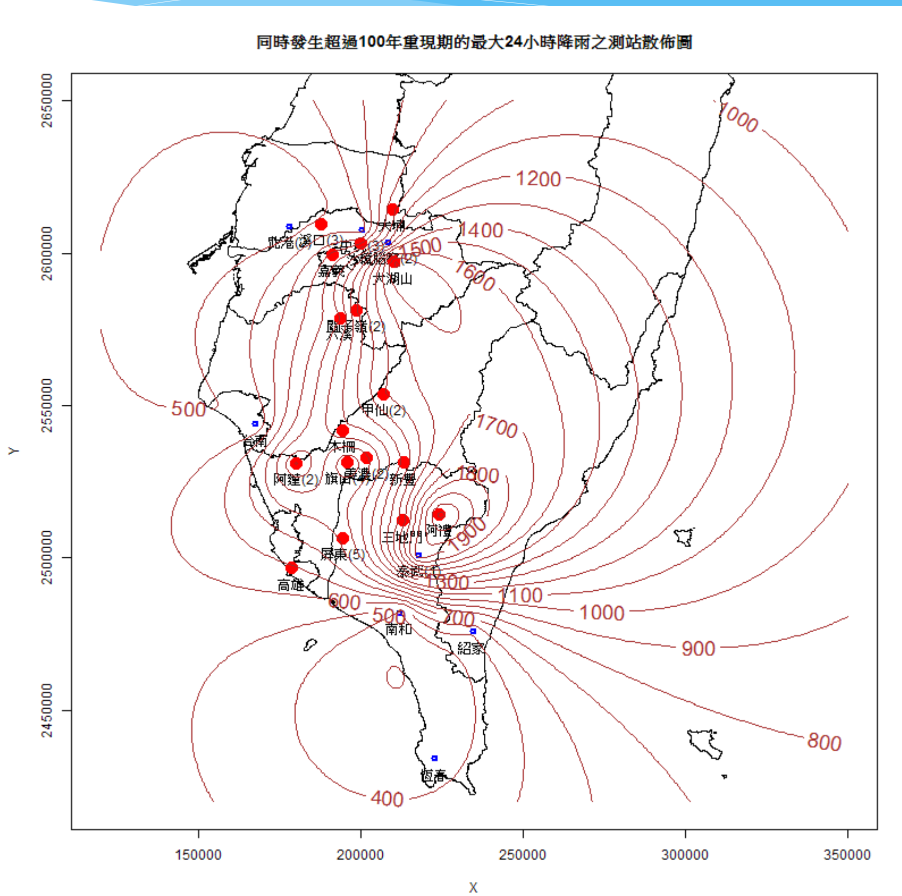
1700
1800
1000
1100
800
700
600
200
400
大埔
中坑(3)
大湖山
甲仙(2)
台南
木柵
阿蓮(2)
新豐
阿禮
屏東(5)
高雄
南和
紹家
恆春

同時發生超過100年重現期的最大24小時降雨之測站散佈圖



Type D


同時發生超過100年重現期的最大24小時降雨之測站數佈圖
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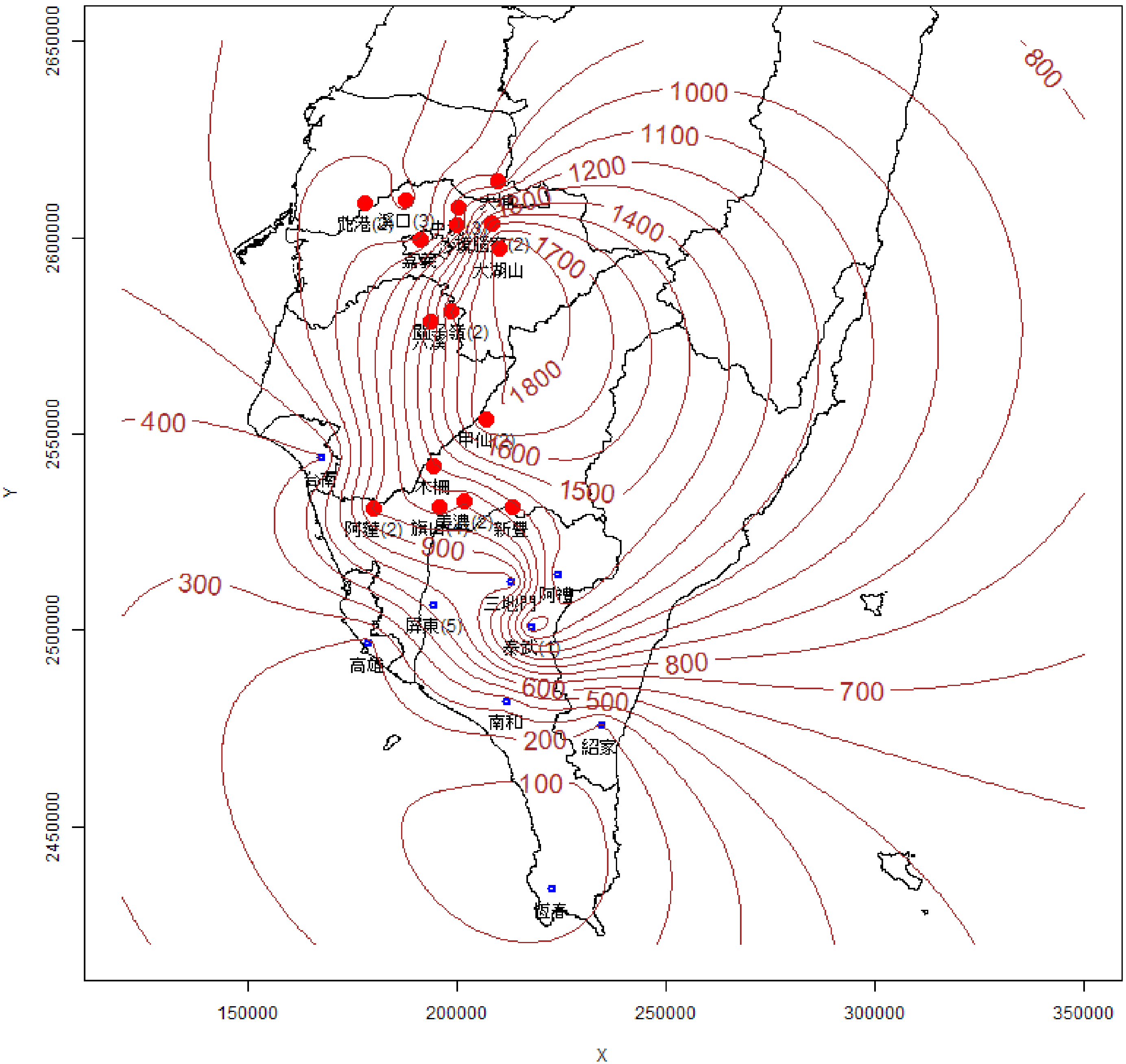
600
500
500
400
甲仙(2)
三地門
阿禮
屏東(5)
高雄
南和
紹家
台南
六龜山
恆春

同時發生超過100年重現期的最大24小時降雨之測站散佈圖



Summary

- We developed a stochastic approach for simulation of multi-site event-max rainfalls to cope with the problems of outliers and short record length in hydrological frequency analysis.
- By increasing the sample size and considering the spatial covariation of EMRs, the return periods of site-specific and multi-site rainfall extremes can be better estimated.

Rationale of BVG simulation using frequency factor

- From the view point of random number generation, the frequency factor can be considered as a random variable *K*, and K_T is a value of *K* with exceedence probability 1/*T*.
- Frequency factor of the Pearson type III distribution can be approximated by

Standard normal deviate

$$K_T \approx z + (z^2 - 1)\frac{\gamma_X}{6} + \frac{1}{3}(z^3 - 6z)\left(\frac{\gamma_X}{6}\right)^2 - (z^2 - 1)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5 \quad \text{[A]}$$

- **General equation for hydrological frequency analysis**

$$\boxed{X_T = \mu_X + K_T \sigma_X}$$

Given μ_X, σ_X and γ_X, if we can generate a set of random numbers of K, say $k_1, k_2, \cdots, k_n$, then a random sample of X, say $x_1, x_2, \cdots, x_n$, can be obtained by $x_i = \mu_X + k_i \sigma_X$.

Note that each $k_i, i = 1,2,\cdots n$, corresponds to its own exceedence probability $1/T_i$.

- **The gamma distribution is a special case of the Pearson type III distribution with a zero location parameter. Therefore, it seems plausible to generate random samples of a bivariate gamma distribution based on two jointly distributed frequency factors.**

$$K_T \approx z + \left(z^2 - 1\right)\frac{\gamma_X}{6} + \frac{1}{3}\left(z^3 - 6z\right)\left(\frac{\gamma_X}{6}\right)^2 - \left(z^2 - 1\right)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5 \qquad \textbf{[A]}$$

Gamma density

$$f_X(x\,;\alpha,\beta)=\frac{1}{\alpha\Gamma(\beta)}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-x/\alpha},\quad 0\le x<+\infty$$

$$\alpha=\frac{\sigma}{\sqrt{\beta}}>0 \qquad \beta=\left(\frac{2}{\gamma}\right)^2>0 \qquad \mu=\alpha\beta=\sigma\sqrt{\beta}>0$$

μ, σ, and γ are the mean, standard deviation, and skewness coefficient of X (or Y), respectively, and α and β are respectively the scale and shape parameters of the gamma density.

$$\sigma=\frac{\mu\gamma}{2}$$

$$K_T \approx z + (z^2 - 1)\frac{\gamma_X}{6} + \frac{1}{3}(z^3 - 6z)\left(\frac{\gamma_X}{6}\right)^2 - (z^2 - 1)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

- **Assume two gamma random variables *X* and *Y* are jointly distributed.**
- **The two random variables are respectively associated with their frequency factors K_X and K_Y.**
- **Equation (A) indicates that the frequency factor K_X of a random variable *X* with gamma density is approximated by a function of the standard normal deviate and the coefficient of skewness of the gamma density.**

$$K_T \approx z + (z^2 - 1)\frac{\gamma_X}{6} + \frac{1}{3}(z^3 - 6z)\left(\frac{\gamma_X}{6}\right)^2 - (z^2 - 1)\left(\frac{\gamma_X}{6}\right)^3 + z\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

Simulation of the frequency factor K_X can be achieved by generating a random sample of the standard normal deviate U, say $u_1, u_2, \cdots, u_n$, and then utilizing Eq. (A) to obtain $k_{x_1}, k_{x_2}, \cdots, k_{x_n}$ from $u_1, u_2, \cdots, u_n$.

However, for a bivariate gamma density $f_{XY}(x, y)$, the two frequency factors K_X and K_Y are correlated through two correlated standard normal deviates U and V, with a correlation coefficient ρ_{UV}.

- **Thus, random number generation of the second frequency factor K_Y must take into consideration the correlation between K_X and K_Y which stems from the correlation between U and V.**

Conditional normal density

- Given a random number of *U*, say *u*, the conditional density of *V* is expressed by the following conditional normal density

$$\phi_{V|U}(v \mid U = u) = \frac{1}{\sqrt{2\pi(1-\rho_{UV}^2)}} \cdot \exp\left\{-\frac{1}{2}\left[\frac{v-\rho_{UV}u}{\sqrt{1-\rho_{UV}^2}}\right]^2\right\}$$

with mean $\rho_{UV}u$ and variance $1-\rho_{UV}^2$.

Thus, based on a random sample $u_1, u_2, \cdots, u_n$ of U, a random sample of V, say $v_1, v_2, \cdots, v_n$, can be generated by a normal random number generator with means $\rho_{UV} u_i \ (i = 1, 2, \cdots, n)$ and variance $1 - \rho_{UV}^2$.

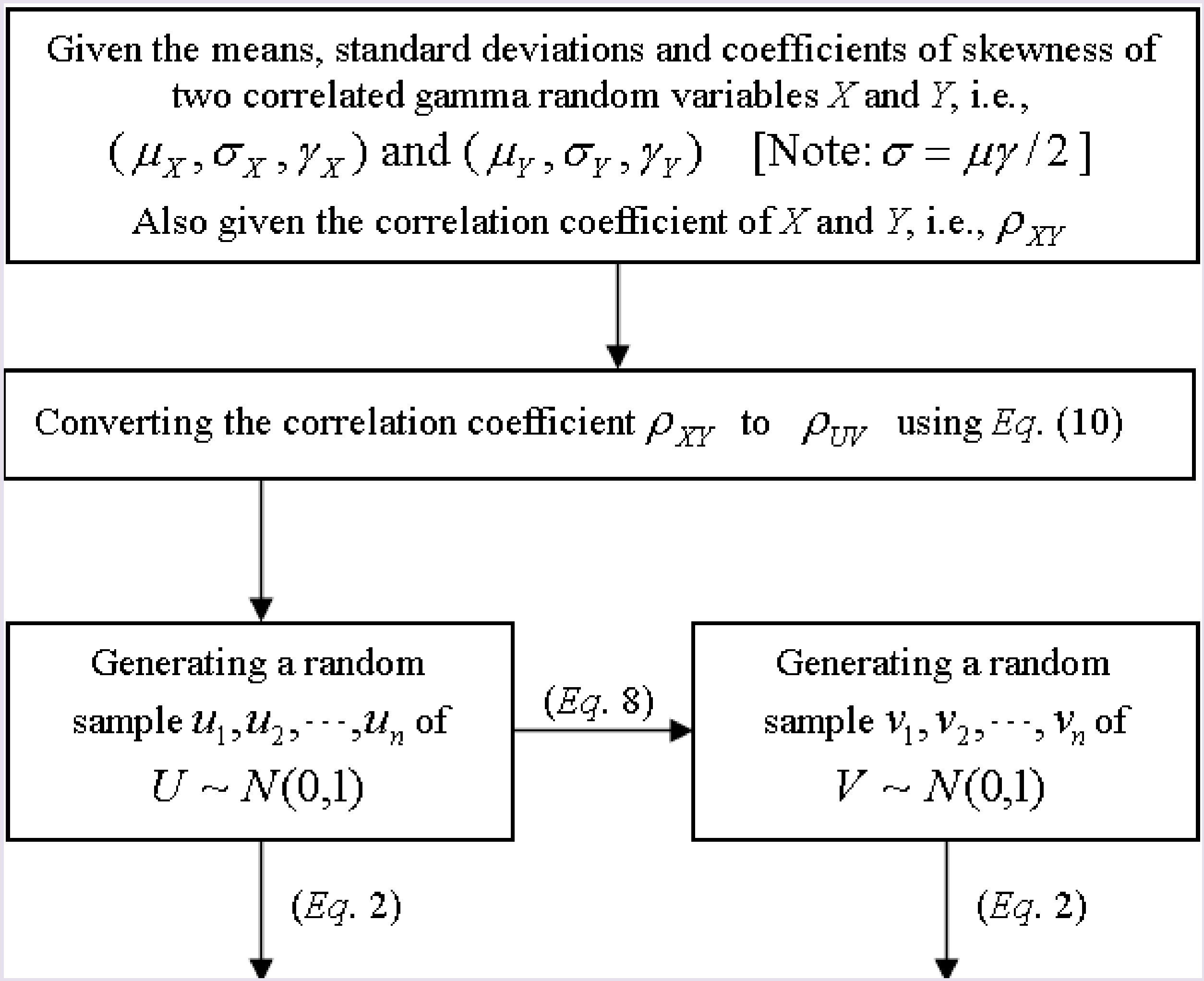
$$\phi_{V|U}(v \mid U = u) = \frac{1}{\sqrt{2\pi(1-\rho_{UV}^2)}} \cdot \exp\left\{ -\frac{1}{2} \left[\frac{v - \rho_{UV} u}{\sqrt{1-\rho_{UV}^2}} \right]^2 \right\}$$

$$X = \mu_X + K\sigma_X$$

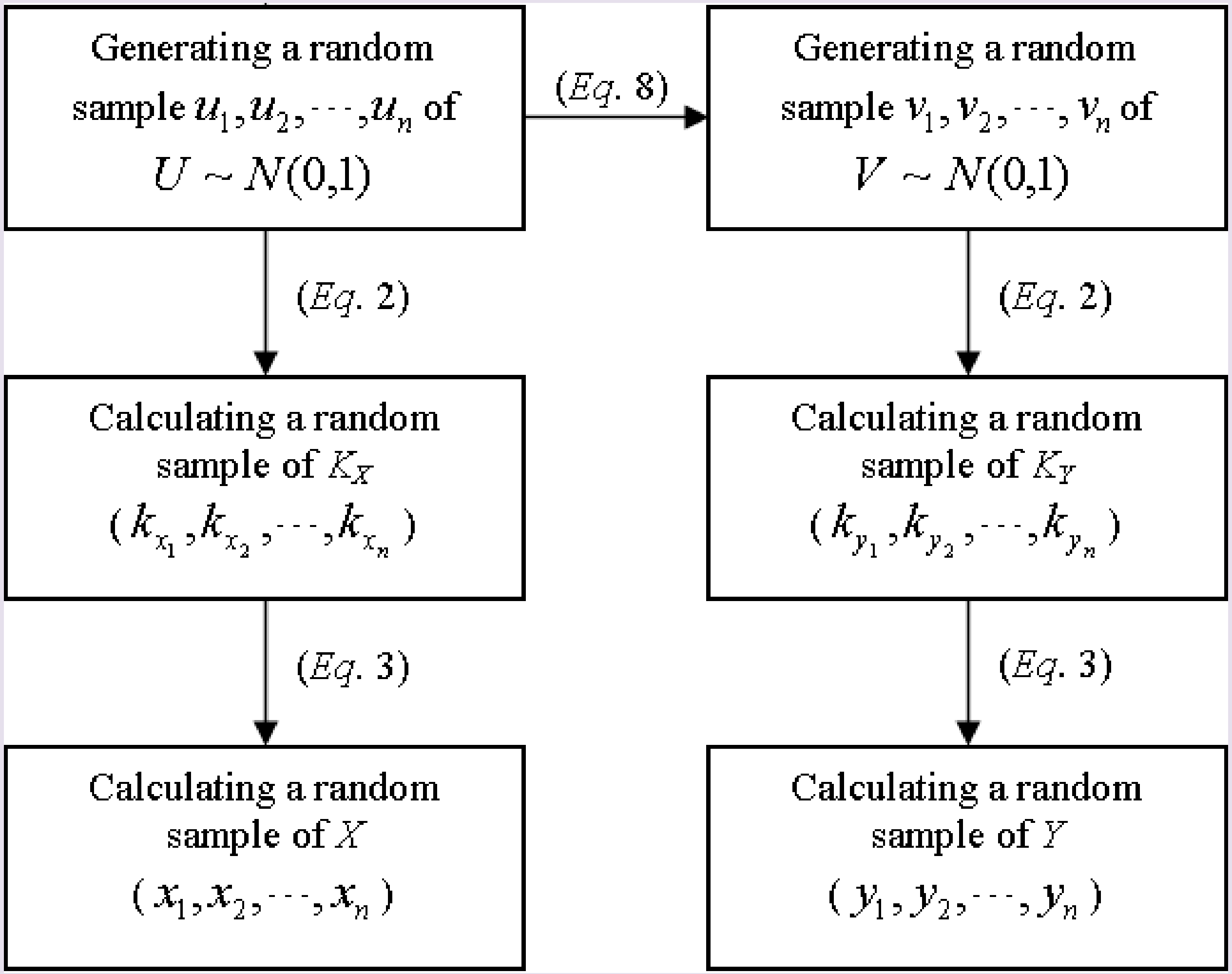
From the two sets of random samples $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$, Eq. (A) can be used to obtain random samples of the two frequency factors K_X and K_Y, i.e., $k_{x_1}, k_{x_2}, \cdots, k_{x_n}$ and $k_{y_1}, k_{y_2}, \cdots, k_{y_n}$.

Given the expected values (μ_X and μ_Y) and standard deviations (σ_X and σ_Y) of random variables X and Y, random samples of the bivariate gamma distribution can be obtained by transferring $k_{x_1}, k_{x_2}, \cdots, k_{x_n}$ and $k_{y_1}, k_{y_2}, \cdots, k_{y_n}$ to $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$.

Flowchart of BVG simulation (1/2)



Flowchart of BVG simulation (2/2)



In practice, stochastic simulation of a bivariate gamma distribution requires the generated random samples to have pre-specified mean, standard deviation, coefficient of skewness ((μ_X, σ_X, γ_X) and (μ_Y, σ_Y, γ_Y)), and correlation coefficient ρ_{XY}. In order for the generated samples to meet such requirements, the correlation coefficient ρ_{UV} must be determined from the pre-specified γ_X, γ_Y, and ρ_{XY} through the following equation:

$\rho_{XY} \sim \rho_{UV}$ Conversion

$$\rho_{XY} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$
$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3$$ **[B]**

$$A_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 \qquad B_X = \frac{\gamma_X}{6} - \left(\frac{\gamma_X}{6}\right)^3 \qquad C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2$$

$$A_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 \qquad B_Y = \frac{\gamma_Y}{6} - \left(\frac{\gamma_Y}{6}\right)^3 \qquad C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2$$

Derivation of the $\rho_{XY} \sim \rho_{UV}$ relationship

Suppose that random variables X and Y form a bivariate gamma distribution. Given the means (μ_X and μ_Y) and standard deviations (σ_X and σ_Y), X and Y can be respectively expressed in terms of their corresponding frequency factors K_X and K_Y , i.e.,

$$X = \mu_X + K_X \sigma_X \quad \text{and} \quad Y = \mu_Y + K_Y \sigma_Y .$$

Note that, with given means μ_X and μ_Y and standard deviations σ_X and σ_Y , the coefficients of skewness γ_X and γ_Y can be readily determined.

From the above equations, it can be easily shown that frequency factors K_X and K_Y are distributed with zero mean and unit standard deviation, and correlation coefficient of X and Y is equivalent to correlation coefficient of K_X and K_Y, i.e.,

$$E[K_X] = E[K_Y] = 0,$$

$$Var[K_X] = Var[K_Y] = 1,$$

and

$$\rho_{XY} = \rho_{K_X K_Y}$$

- **Frequency factors K_X and K_Y can be respectively approximated by**

$$K_X \approx U + \left(U^2 - 1\right)\frac{\gamma_X}{6} + \frac{1}{3}\left(U^3 - 6U\right)\left(\frac{\gamma_X}{6}\right)^2 - \left(U^2 - 1\right)\left(\frac{\gamma_X}{6}\right)^3 + U\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

$$K_Y \approx V + \left(V^2 - 1\right)\frac{\gamma_Y}{6} + \frac{1}{3}\left(V^3 - 6V\right)\left(\frac{\gamma_Y}{6}\right)^2 - \left(V^2 - 1\right)\left(\frac{\gamma_Y}{6}\right)^3 + V\left(\frac{\gamma_Y}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5$$

where U and V both are random variables with standard normal density and are correlated with correlation coefficient ρ_{UV}.

- **Correlation coefficient of K_X and K_Y can be derived as follows:**

$$\rho_{K_X K_Y} = Cov(K_X, K_Y) = E[K_X K_Y]$$

$$\approx E\left\{ \begin{aligned} &\left[U + (U^2-1)\frac{\gamma_X}{6} + \frac{1}{3}(U^3-6U)\left(\frac{\gamma_X}{6}\right)^2 - (U^2-1)\left(\frac{\gamma_X}{6}\right)^3 + U\left(\frac{\gamma_X}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5 \right] \cdot \\ &\left[V + (V^2-1)\frac{\gamma_Y}{6} + \frac{1}{3}(V^3-6V)\left(\frac{\gamma_Y}{6}\right)^2 - (V^2-1)\left(\frac{\gamma_Y}{6}\right)^3 + V\left(\frac{\gamma_Y}{6}\right)^4 - \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5 \right] \end{aligned} \right\}$$

$$K_X \approx U\left[1+\left(\frac{\gamma_X}{6}\right)^4\right]+\left(U^2-1\right)\left[\frac{\gamma_X}{6}-\left(\frac{\gamma_X}{6}\right)^3\right]+\frac{1}{3}\left(U^3-6U\right)\left(\frac{\gamma_X}{6}\right)^2-\frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

$$= A_X U + B_X\left(U^2-1\right)+C_X\left(U^3-6U\right)+D_X$$

$$K_Y \approx V\left[1+\left(\frac{\gamma_Y}{6}\right)^4\right]+\left(V^2-1\right)\left[\frac{\gamma_Y}{6}-\left(\frac{\gamma_Y}{6}\right)^3\right]+\frac{1}{3}\left(V^3-6V\right)\left(\frac{\gamma_Y}{6}\right)^2-\frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5$$

$$= A_Y V + B_Y\left(V^2-1\right)+C_Y\left(V^3-6V\right)+D_Y$$

$$A_X = 1+\left(\frac{\gamma_X}{6}\right)^4,\ B_X = \frac{\gamma_X}{6}-\left(\frac{\gamma_X}{6}\right)^3,\ C_X = \frac{1}{3}\left(\frac{\gamma_X}{6}\right)^2,\ D_X = -\frac{1}{3}\left(\frac{\gamma_X}{6}\right)^5$$

$$A_Y = 1+\left(\frac{\gamma_Y}{6}\right)^4,\ B_Y = \frac{\gamma_Y}{6}-\left(\frac{\gamma_Y}{6}\right)^3,\ C_Y = \frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^2,\ D_Y = -\frac{1}{3}\left(\frac{\gamma_Y}{6}\right)^5$$

$$E\left[K_X K_Y\right]$$

$$\approx E\left[\begin{array}{l}
A_X A_Y UV + \cancel{A_X B_Y U\left(V^2-1\right)} + A_X C_Y U\left(V^3-6V\right) \\
+\cancel{A_X D_Y U} + \cancel{B_X A_Y V\left(U^2-1\right)} + B_X B_Y\left(U^2-1\right)\left(V^2-1\right) \\
+\cancel{B_X C_Y\left(U^2-1\right)\left(V^3-6V\right)} + \cancel{B_X D_Y\left(U^2-1\right)} \\
+C_X A_Y\left(U^3-6U\right)V + \cancel{C_X B_Y\left(U^3-6U\right)\left(V^2-1\right)} \\
+C_X C_Y\left(U^3-6U\right)\left(V^3-6V\right) + \cancel{C_X D_Y\left(U^3-6U\right)} \\
+\cancel{D_X A_Y V} + \cancel{D_X B_Y\left(V^2-1\right)} + \cancel{D_X C_Y\left(V^3-6V\right)} + D_X D_Y
\end{array}\right]$$

$$E\left[K_X K_Y\right]$$

$$\approx E\begin{bmatrix} A_X A_Y UV + A_X C_Y U\left(V^3 - 6V\right) + B_X B_Y\left(U^2 - 1\right)\left(V^2 - 1\right) \\ + C_X A_Y\left(U^3 - 6U\right)V + C_X C_Y\left(U^3 - 6U\right)\left(V^3 - 6V\right) + D_X D_Y \end{bmatrix}$$

$$E[K_X] \approx E\left[A_X U + B_X\left(U^2 - 1\right) + C_X\left(U^3 - 6U\right) + D_X\right] = D_X$$

Since K_X and K_Y are distributed with zero means, it follows that

$$D_X = D_Y \approx 0$$

$$\rho_{K_X K_Y} = E\left[K_X K_Y\right]$$

$$\approx E\left[\begin{array}{l} A_X A_Y UV + A_X C_Y U\left(V^3 - 6V\right) + B_X B_Y\left(U^2 - 1\right)\left(V^2 - 1\right) \\ + C_X A_Y\left(U^3 - 6U\right) V + C_X C_Y\left(U^3 - 6U\right)\left(V^3 - 6V\right) \end{array}\right]$$

$$= A_X A_Y \rho_{UV} + A_X C_Y\left[E\left(UV^3\right) - 6\rho_{UV}\right] + B_X B_Y\left[E\left(U^2V^2\right) - 1\right]$$

$$+ C_X A_Y\left[E\left(U^3V\right) - 6\rho_{UV}\right]$$

$$+ C_X C_Y\left[E\left(U^3V^3\right) - 6E\left(UV^3\right) - 6E\left(U^3V\right) + 36\rho_{UV}\right]$$

- **It can also be shown that**

$$E\left(U^2V^2\right) = 2\rho_{UV}^2 + 1 \qquad E\left(U^3V^3\right) = 6\rho_{UV}^3 + 9\rho_{UV}$$

$$E\left(U^3V\right) = E\left(UV^3\right) = 3\rho_{UV}$$

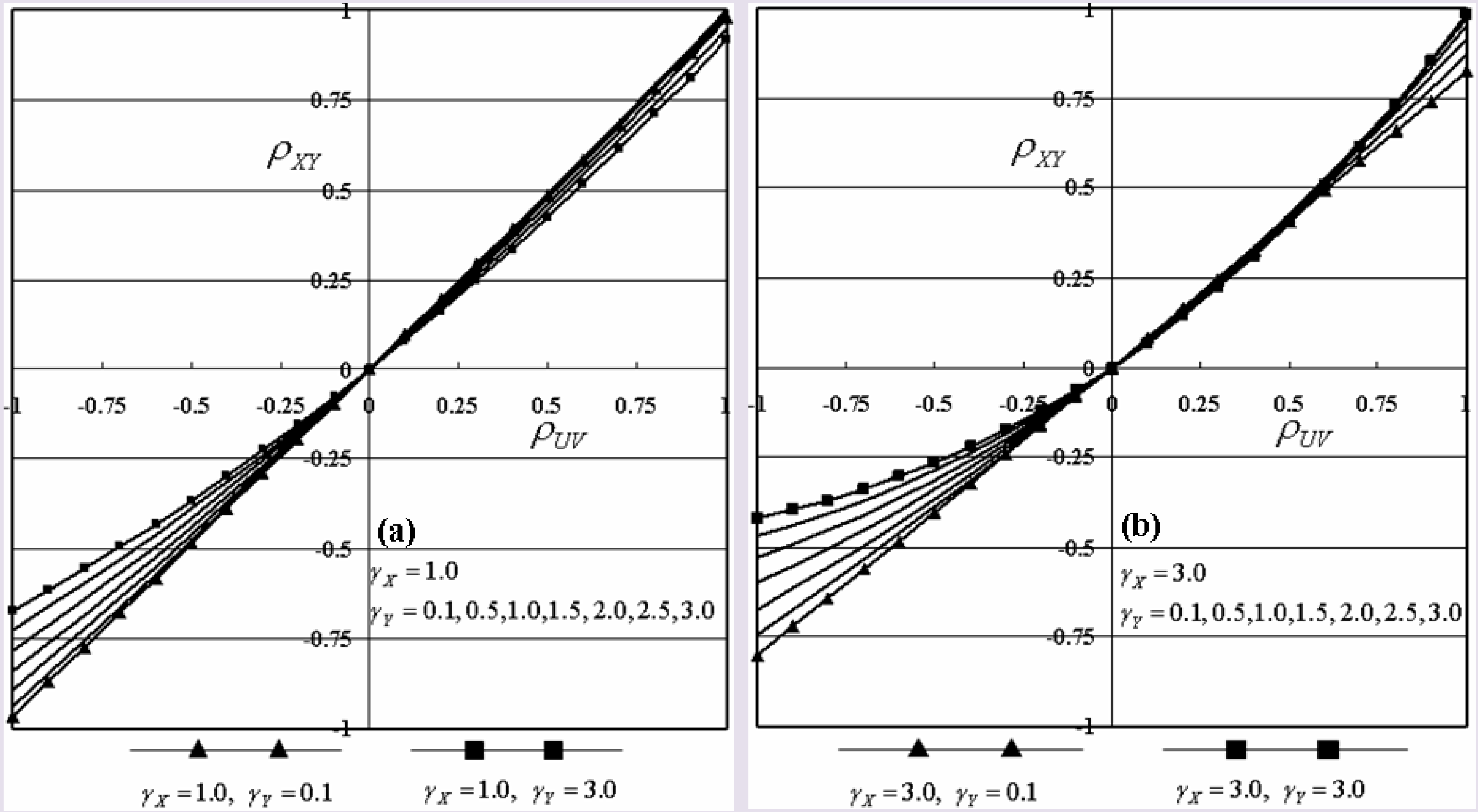
Thus,

$$\rho_{XY} = \rho_{K_X K_Y} \approx \left(A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y\right)\rho_{UV}$$

$$+ 2B_X B_Y \rho_{UV}^2 + 6C_X C_Y \rho_{UV}^3$$

Equation (B) indicates that ρ_{XY} can be expressed as a third order polynomial of ρ_{UV} . It is therefore of practical concern whether there exits a unique ρ_{UV} for a given set of (γ_X , γ_Y , ρ_{XY}). Or equivalently, given a set of (γ_X , γ_Y , ρ_{XY}), does Eq. (B) return a single-value of ρ_{UV} ?

$\rho_{XY} \sim \rho_{UV}$ Single - Value Relationship



We have also proved that Eq. (B) is indeed a single-value function.

Proof of Eq. (B) as a single-value function

Let $f(\rho_{UV}) = \partial\rho_{XY} / \partial\rho_{UV}$. From Eq. (B) we have

$$f(\rho_{UV}) = (A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y)$$
$$+ 4B_X B_Y \rho_{UV} + 18C_X C_Y \rho_{UV}^2$$

$$A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y = (A_X - 3C_X)(A_Y - 3C_Y)$$

$$A_X - 3C_X = 1 + \left(\frac{\gamma_X}{6}\right)^4 - \left(\frac{\gamma_X}{6}\right)^2 = \left[\left(\frac{\gamma_X}{6}\right)^2 - 1\right]^2 + \left(\frac{\gamma_X}{6}\right)^2 > 0$$

$$A_Y - 3C_Y = 1 + \left(\frac{\gamma_Y}{6}\right)^4 - \left(\frac{\gamma_Y}{6}\right)^2 = \left[\left(\frac{\gamma_Y}{6}\right)^2 - 1\right]^2 + \left(\frac{\gamma_Y}{6}\right)^2 > 0 .$$

- **Therefore,**

$$A_X A_Y - 3A_X C_Y - 3C_X A_Y + 9C_X C_Y$$

$$= \left\{ \left[\left(\frac{\gamma_X}{6} \right)^2 - 1 \right]^2 + \left(\frac{\gamma_X}{6} \right)^2 \right\} \left\{ \left[\left(\frac{\gamma_Y}{6} \right)^2 - 1 \right]^2 + \left(\frac{\gamma_Y}{6} \right)^2 \right\}$$

Let $g(\rho_{UV}) = 4B_X B_Y \rho_{UV} + 18C_X C_Y \rho_{UV}^2$

$$= \frac{\gamma_X \gamma_Y}{9}\left[\left(\frac{\gamma_X}{6}\right)^2 - 1\right]\left[\left(\frac{\gamma_Y}{6}\right)^2 - 1\right]\rho_{UV} + \frac{18}{9}\left(\frac{\gamma_X}{6}\right)^2\left(\frac{\gamma_Y}{6}\right)^2 \rho_{UV}^2$$

Also, let $G_X = \frac{\gamma_X}{6}$, $G_Y = \frac{\gamma_Y}{6}$. We then have

$$g(\rho_{UV}) = 4G_X G_Y (G_X^2 - 1)(G_Y^2 - 1)\rho_{UV} + 2G_X^2 G_Y^2 \rho_{UV}^2$$

$$f(\rho_{UV})=\left[(G_X^2-1)^2+G_X^2\right]\left[(G_Y^2-1)^2+G_Y^2\right]$$
$$+4G_XG_Y(G_X^2-1)(G_Y^2-1)\rho_{UV}+2G_X^2G_Y^2\rho_{UV}^2$$

$$\frac{\partial f}{\partial \rho_{UV}}=4G_XG_Y(G_X^2-1)(G_Y^2-1)+4G_X^2G_Y^2\rho_{UV}$$

Let $\dfrac{\partial f}{\partial \rho_{UV}}=0$, it yields an extreme value of f at

$$\rho_{UV}^{*}=-\frac{(G_X^2-1)(G_Y^2-1)}{G_XG_Y}$$

$$f(\rho_{UV}^{*}) = \left[(G_X^2-1)^2+G_X^2\right]\left[(G_Y^2-1)^2+G_Y^2\right]$$
$$-4(G_X^2-1)^2(G_Y^2-1)^2+2(G_X^2-1)^2(G_Y^2-1)^2$$

$$=\left[(G_X^2-1)^2+G_X^2\right]\left[(G_Y^2-1)^2+G_Y^2\right]-2(G_X^2-1)^2(G_Y^2-1)^2$$

$$=G_X^2(G_Y^2-1)^2+(G_X^2-1)^2G_Y^2+G_X^2G_Y^2-(G_X^2-1)^2(G_Y^2-1)^2$$

Since $-1 \le \rho_{UV} \le 1$ (or equivalently, $\rho_{UV}^2 \le 1$), it yields

$$(G_X^2-1)^2(G_Y^2-1)^2 \le G_X^2G_Y^2$$

Thus,

$$f(\rho_{UV}^{*}) \ge G_X^2(G_Y^2-1)^2+(G_X^2-1)^2G_Y^2 > 0$$

We now check the second derivative of $f(\rho_{UV})$, i.e.,

$$\frac{\partial^2 f}{\partial(\rho_{UV})^2} = 4G_X^2 G_Y^2 > 0$$

Therefore, $f(\rho_{UV}^*) > 0$ is the minimum of the function $f(\rho_{UV}) = \partial\rho_{XY} / \partial\rho_{UV}$. It follows that

$$f(\rho_{UV}) = \partial\rho_{XY} / \partial\rho_{UV} > 0$$

for all possible values of ρ_{UV} .

- **The above equation indicates ρ_{XY} increases with increasing ρ_{UV} , and thus Eq. (B) is a single-value function.**

Conceptual description of a gamma random field simulation approach

Given a pair of bivariate gamma random variables (X,Y) with know properties $\mu_X, \mu_Y, \gamma_X, \gamma_Y, \rho_{XY}$

↓

Converting ρ_{XY} to ρ_{UV} where (U,V) represents a pair of bivariate standard normal variables.

↓

Given a homogeneous and isotropic random field $Z(x)$ with known gamma density and covariance function $C_Z(h)$ or variogram $\gamma_Z(h)$.

↓

Converting $C_Z(h)$ to $C_W(h)$ where $W(x)$ is a random field with standard normal density and covariance function $C_W(h)$.

↓

Generating a random sample of (U,V) with sample size n, i.e. $\{(u_i, v_i), i = 1, \cdots, n\}$.

Individually and independently converting u_i to x_i and v_i to y_i. The resultant $\{(x_i, y_i), i = 1, \cdots, n\}$ is a random sample of the bivariate random variables (X,Y).

Generating a realization of W, i.e. $\{w(i, j), i = 1, \cdots, n\ ;\ j = 1, \cdots, m\}$ where (i,j) represents a spatial location and n and m defines the extent of the spatial domain.

Individually and independently converting $w(i, j)$ to $z(i, j)$. The resultant $\{z(i, j), i = 1, \cdots, n\ ;\ j = 1, \cdots, m\}$ is a realization of the random field $Z(x)$.