# Numerical Partial Differential Equations:

## Pseudospectral Methods for Wave Equations on Spherical Surfaces

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# Outline

# Legendre Pseudospectral Method I

## Concepts and Notations

- $P_N(x)$: Legendre polynomial of degree $N$.
- Legendre-Gauss-Lobatto (LGL) grid points $x_i$

$$-1 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1, \quad \text{roots of } (1-x^2)P_N'(x)$$

- Lagrange interpolation polynomials:

$$l_j(x) = \frac{-(1-x^2)P_N'(x)}{N(N+1)(x-x_j)P_N(x_j)}, \quad l_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Approximation of $u(x)$ defined on $[-1,1]$ and its derivative $u'$:

$$u(x) \approx \mathcal{I}_N u(x) = \sum_{j=0}^{N} l_j(x)u(x_j), \quad u'(x) \approx \frac{d}{dx}\mathcal{I}_N u(x) = \sum_{j=0}^{N} l_j'(x)u(x_j)$$

- LGL quadrature integration rule:

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=0}^{N} \omega_i f(x_i), \quad \omega_i\text{: quadrature weights}$$

provided that $f(x)$ is a polynomial of degree at most $2N-1$.

# Legendre Pseudospectral Method II

## Exponential Convergency

- Numerical derivatives at the LGL points:

$$u'(x_i) \approx \frac{d}{dx}\mathcal{I}_N u(x_i) = \sum_{j=0}^{N} l_j'(x_i)u(x_j)$$

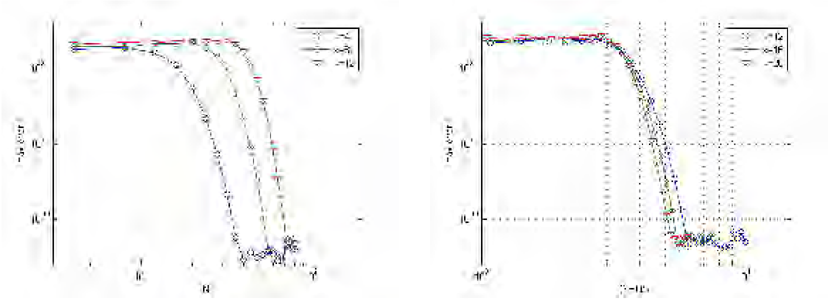

Figure: Errors of the numerical differentiation $u'(x_i) - \frac{d}{dx}\mathcal{I}_N u(x_i)$ for $u = \sin(k\pi x)$ for various values of $k$.

# Model Wave Problem

Consider $u(x,t)$ satisfying the problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad x \in [-1,1] \quad t \geq 0$$
$$u(x,0) = f(x) \quad x \in [-1,1] \quad t = 0$$
$$u(-1,t) = g(t) \quad x = -1 \quad t \geq 0.$$

Define the energy-norm for $u$ as

$$E(t) = \int_{-1}^{1} u^2(x,t)\,dx$$

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Then we have

$$\frac{dE(t)}{dt} = \frac{d}{dt}\int_{-1}^{1} u^2 dx = \int_{-1}^{1} 2u\frac{\partial u}{\partial t}\,dx = -\int_{-1}^{1} 2u\frac{\partial u}{\partial x}\,dx = -\int_{-1}^{1} \frac{\partial u^2}{\partial x}\,dx$$
$$= u^2(-1,t) - u^2(1,t) = g^2(t) - u^2(1,t) \leq g^2(t)$$

implying that

$$E(t) \leq E(0) + \int_0^t g^2(\xi)\,d\xi \leq E(0) + t\cdot G, \quad G = \max_{\xi\in[0,t]} g^2(\xi)$$
$$\Rightarrow \int_{-1}^{1} u^2(x,t)dx \leq \int_{-1}^{1} f^2(x)dx + t\cdot G$$

The solution is bounded by the prescribed data.

# Pseudospectral Penalty Method for Wave Problem I

Consider the problem:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad x \in [-1,1],\ t \geq 0$$
$$u(x,0) = f(x) \quad x \in [-1,1]$$
$$u(-1,t) = g(t) \quad t \geq 0$$

satisfying the energy estimate

$$\int_{-1}^{1} u^2(x,t)dx \leq \int_{-1}^{1} f^2(x)dx + t \cdot G$$

$$G = \max_{\xi \in [0,t]} g^2(\xi)$$

We seek a numerical solution of the form

$$v(x,t) = \sum_{j=0}^{N} l_j(x) v_j(t) \quad \text{(polynomial of degree } N \text{ in } x\text{)}$$

satisfying the collocation equations:

$$\frac{\partial v(x_i,t)}{\partial t} + \frac{\partial v(x_i,t)}{\partial x} = -\tau \cdot \delta_{0i} \cdot (v_0(t) - g(t)), \quad v_i(0) = f(x_i), \quad i = 0, 1, \cdots, N$$

where the boundary condition is imposed weakly and $\tau$ is a parameter.

# Pseudospectral Penalty Method for Wave Problem II

Semi-discrete scheme:

$$\frac{\partial v(x_i,t)}{\partial t} + \frac{\partial v(x_i,t)}{\partial x} = -\tau\,\delta_{0i}\,(v(-1,t) - g(t)), \quad \forall i = 0,1,2,\ldots,N$$

Observe that[1]

1. Consistency: Replacing $v(x_i,t)$ by $u(x_i,t)$ we obtain

$$\frac{\partial u(x_i,t)}{\partial t} + \frac{\partial u(x_i,t)}{\partial x} = 0, \quad \forall i = 0,1,2,\ldots,N \quad \text{(recover PDE)}$$

   independent of $\tau$.

2. as $\tau \to 0$

$$\frac{\partial v(x_0,t)}{\partial t} + \frac{\partial v(x_0,t)}{\partial x} = -\tau\,(v(-1,t) - g(t)) \to 0 \quad \text{(mimicking PDE)}$$

3. as $\tau \to \infty$

$$v(-1,t) - g(t) = \frac{-1}{\tau}\left(\frac{\partial v(x_0,t)}{\partial t} + \frac{\partial v(x_0,t)}{\partial x}\right) \to 0 \quad \text{(mimicking BC)}$$

[1] D. Funaro and D. Gottlieb, A new method of imposing boundary conditions in pseudospectral approximations of hyperbolic equations, Math. Comp., 51 (1988) 599-613.

# Pseudospectral Penalty Method for Wave Problem III

## Numerical Energy Estimate

We have the scheme:

$$\frac{\partial v(x_i,t)}{\partial t} + \frac{\partial v(x_i,t)}{\partial x} = -\tau\, \delta_{0i}\,(v_0 - g(t)), \quad \forall i = 0,1,2,\ldots,N$$

Define the discrete energy-norm as

$$E_D(t) = \sum_{i=0}^{N} v_j^2(t)\omega_i = \sum_{i=0}^{N} v^2(x_i,t)\omega_i \quad \text{(mimicking} \quad E = \int_{-1}^{1} u^2(x,t)dx)$$

We have the energy rate equation as

$$\begin{aligned}
\frac{dE_D(t)}{dt} &= \sum_{i=0}^{N} 2v_i \frac{dv_i}{dt}\omega_i = -\sum_{i=0}^{N} 2v(x_i,t)\frac{\partial v(x_i,t)}{\partial x}\,\omega_i - \sum_{i=0}^{N} 2\tau\,\delta_{0i}\,\omega_i\, v_i\,(v_0 - g(t)) \\
&= \int_{-1}^{1} 2v(x,t)\frac{\partial v(x_i,t)}{\partial x}\,dx - 2\tau\omega_0 v_0(v_0 - g(t)) \\
&= -v(x,t)|_{-1}^{1} - 2\tau\omega_0 v_0(v_0 - g(t)) = -v_N^2 + v_0^2 - 2\tau\omega_0 v_0(v_0 - g(t))
\end{aligned}$$

# Pseudospectral Penalty Method for Wave Problem IV

We obtain the energy rate equation as

$$\frac{dE_D(t)}{dt} = -v_N^2 + v_0^2 - 2\tau\omega_0 v_0 (v_0 - g(t))$$

Taking $\tau = 1/\omega_0$ we obtain

$$\begin{aligned}\frac{dE_D(t)}{dt} &= -v_N^2 + \underline{v_0^2 - 2v_0^2 + 2v_0 g(t) - g^2(t)} + g^2(t) \\ &= -v_N^2 - \underline{(v_0 - g(t))^2} + g^2(t) \le g^2(t)\end{aligned}$$

implying

$$E_D(t) \le E_D(0) + \int_0^t g^2(\xi)\, d\xi \le E_D(0) + t\, G, \quad G = \max_{\xi \in [0,t]} g^2(\xi)$$

or equivalently,

$$\sum_{i=0}^{N} v_j^2(t)\omega_i \le \sum_{i=0}^{N} f_j^2 \omega_i + t\, G \quad \text{mimicking} \quad \int_{-1}^{1} u^2(x,t)dx \le \int_{-1}^{1} f^2(x)dx + t\cdot G.$$

The numerical solution is bounded by the prescribed data independent of $N$, implying stability.

# Numerical Computations

Semi-discrete scheme:

$$\frac{\partial v(x_i,t)}{\partial t} + \frac{\partial v(x_i,t)}{\partial x} = -\tau\,\delta_{0i}\,(v(-1,t) - g(t)), \quad \forall i = 0,1,2,\dots,N$$
$$v(x_i,0) = f(x_i)$$

Introducing the following notations

$$\boldsymbol{v}(t) = \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_N \end{bmatrix}, \boldsymbol{f} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix}, \boldsymbol{e}_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} l_0'(x_0) & l_1'(x_0) & \cdots & l_N'(x_0) \\ l_0'(x_1) & l_1'(x_1) & \cdots & l_N'(x_1) \\ \vdots & & & \vdots \\ l_0'(x_N) & l_1'(x_N) & \cdots & l_N'(x_N) \end{bmatrix}$$

We can express the scheme as a system of ODEs:

$$\frac{d\boldsymbol{v}(t)}{dt} = -\boldsymbol{D}\boldsymbol{v} - \tau\boldsymbol{e}_0(v_0(t) - g(t))$$
$$\boldsymbol{v}(0) = \boldsymbol{f}$$

which can be solved by ODE integration methods, for example, Runge-Kutta methods.

# Convergence Test

Test example:

$$u(x,t) = \sin(\pi(x-t))$$
$$u(x,0) = \sin(\pi x)$$
$$u(-1,t) = \sin(\pi(-1-t))$$

Legendre method in space

Runge-Kutta 4th order in time

| N | Error | Order |
|---|---|---|
| 8 | 5.3860e-03 | — |
| 12 | 7.7090e-06 | 16.15 |
| 16 | 2.4318e-06 | 4.01 |
| 20 | 9.9604e-07 | 4.00 |
| 24 | 4.8032e-07 | 4.00 |

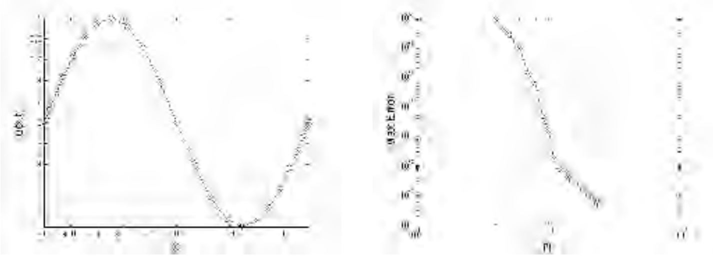


Figure: Left: Computed wave profile at time = 1.00. Right: Maximum errors $|u(x,t) - v_j(t)|$ for different values of $N$ at time = 1.00.

# Instability and Penalty Boundary Conditions

Scheme:

$$\frac{d\boldsymbol{v}(t)}{dt} = -\boldsymbol{D}\boldsymbol{v} - \tau \boldsymbol{e}_0 (v_0(t) - g(t)), \quad \boldsymbol{v}(0) = \boldsymbol{f}$$

For stability we need $\tau \geq 0.5/\omega_0$.

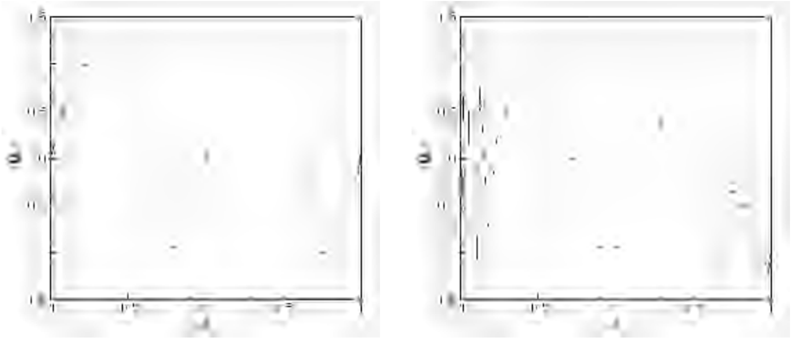
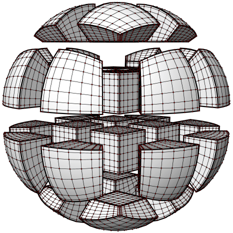
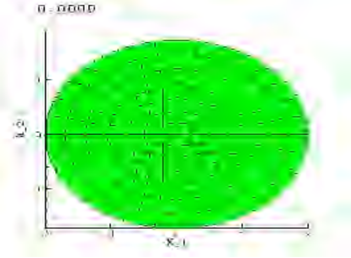


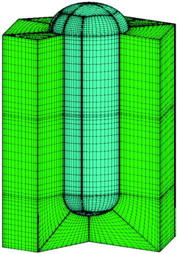
Figure: Left: Solution profile obtained by $\tau = 0.5/\omega_0$.Right: Solution profile obtained by $\tau = 0.47/\omega_0$

# Multidomian Pseudospectral Penalty Formulation

- Multidomian formulation
- Complex Physics Problems
  - Elastodynamics
  - Fluid dynamics
  - Electromagnetics







# General Runge-Kutta (RK) Methods

All discretizations of the spatial operator in a PDE, result in a system of ODEs of the form

$$\frac{du_N}{dt} = \mathcal{L}_N(u_N(x,t),\, x,\, t),$$

or in matrix-vector form

$$\frac{d\boldsymbol{u}(t)}{dt} = \mathrm{L}(\boldsymbol{u}(t),t), \quad \boldsymbol{u} = [u_0(t)\, u_1(t)\, ....\, u_N(t)]^T,$$

$\mathrm{L}$ : matrix representation of $\mathcal{L}_N$.

$s$-stage explicit Runge-Kutta scheme:

$$\boldsymbol{k}_1 = \mathrm{L}(\boldsymbol{u}^n,\, n\Delta t), \quad \boldsymbol{u}^n = \boldsymbol{u}(t_n)$$

$$\boldsymbol{k}_i = \mathrm{L}\left(\boldsymbol{u}^n + \Delta t \sum_{j=1}^{i-1} a_{ij}\boldsymbol{k}_i,\ (n+c_i)\Delta t\right)$$

$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^n + \Delta t \sum_{i=1}^{s} b_i \boldsymbol{k}_i$$

The choice of the constants $a_{ij},\ c_i$ and $b_i$ determines the accuracy and efficiency of the overall scheme.

# Fourth-Order Four-Stage RK Methods

The classical fourth-order accurate, four-stage scheme is

$$
\begin{aligned}
\boldsymbol{k}_1 &= \mathrm{L}(\boldsymbol{u}^n,\, n\Delta t) \\
\boldsymbol{k}_2 &= \mathrm{L}\left(\boldsymbol{u}^n + \frac{1}{2}\Delta t\,\boldsymbol{k}_1,\ (n+\frac{1}{2})\Delta t\right) \\
\boldsymbol{k}_3 &= \mathrm{L}\left(\boldsymbol{u}^n + \frac{1}{2}\Delta t\,\boldsymbol{k}_2,\ (n+\frac{1}{2})\Delta t\right) \\
\boldsymbol{k}_4 &= \mathrm{L}\left(\boldsymbol{u}^n + \Delta t\,\boldsymbol{k}_3,\ (n+1)\Delta t\right) \\
\boldsymbol{u}^{n+1} &= \boldsymbol{u}^n + \frac{\Delta t}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4).
\end{aligned}
$$

- The Runge-Kutta schemes require more evaluations of $\mathrm{L}$ then the multi-step schemes require to advance a time-step.
- Unlike the multi-step schemes, the Runge-Kutta methods require no information from previous time-steps.

# Satbility Region of RK Methods I

To study the linear stability of these schemes we consider the linear scalar equation

$$\frac{du}{dt} = \lambda u,$$

The general s-stage scheme can be expressed as a truncated Taylor expansion of the exponential functions

$$u^{n+1} = \sum_{i=1}^{s} \frac{(\lambda \Delta t)^i}{i!} u^n.$$

Thus, the scheme is stable for a region of $\lambda \Delta t$ for which

$$\left| \sum_{i=1}^{s} \frac{(\lambda \Delta t)^i}{i!} \right| \leq 1.$$

# Stability Region of RK Methods II

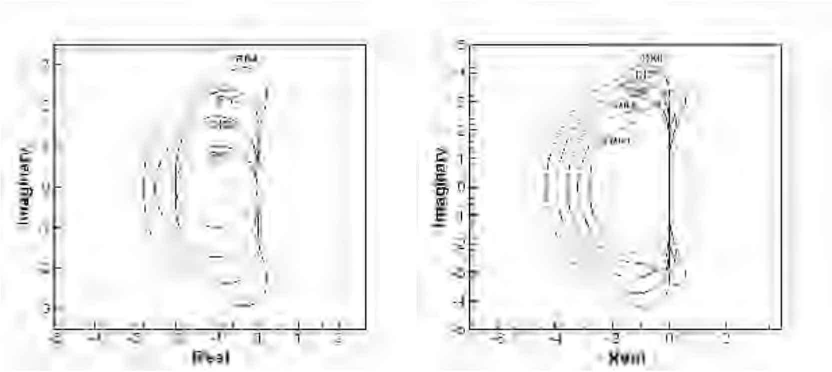



Figure: Stability regions for Runge-Kutta methods

Necessary condition for stability: All the eigenvalues of $\mathrm{L}(\boldsymbol{u}, t) \cdot \Delta t$ must lie in the stability region for a given method.

# Low-Storage RK Methods (RK3S4)

The **s-stage low-storage method** is

$$\boldsymbol{u}_0 = \boldsymbol{u}^n$$
$$\forall j \in [1, \ldots, s] : \begin{cases} \boldsymbol{k}_j = a_j \boldsymbol{k}_{j-1} + \Delta t \, \mathrm{L}\big(\boldsymbol{u}_j, \, (n + c_j)\Delta t\big) \\ \boldsymbol{u}_j = \boldsymbol{u}_{j-1} + b_j \boldsymbol{k}_j \end{cases}$$
$$\boldsymbol{u}^{n+1} = \boldsymbol{u}_s$$

- $a_j, \; b_j$ and $c_j$ are chosen to yield a scheme of order $s - 1$.
- For the scheme to be self-starting we require that $a_1 = 0$.
- We need only two storage levels containing $\boldsymbol{k}_j$ and $\boldsymbol{u}_j$ to advance the solution.

A four-stage third-order RK scheme is obtained using the constants

$$a_1 = 0 \quad b_1 = \tfrac{1}{3} \quad c_1 = 0 \quad a_2 = -\tfrac{11}{15} \quad b_2 = \tfrac{5}{6} \quad c_2 = \tfrac{1}{3}$$
$$a_3 = -\tfrac{5}{3} \quad b_3 = \tfrac{3}{5} \quad c_3 = \tfrac{5}{9} \quad a_4 = -1 \quad b_4 = \tfrac{1}{4} \quad c_4 = \tfrac{8}{9}.$$

# Low-Storage RK Methods (RK4S5)

The **s-stage low-storage method** is

$$\begin{aligned}
&\boldsymbol{u}_0 = \boldsymbol{u}^n \\
\forall j \in [1,\dots,s]: \quad &\begin{cases} \boldsymbol{k}_j = a_j \boldsymbol{k}_{j-1} + \Delta t \, \mathrm{L}\big(\boldsymbol{u}_j,\ (n+c_j)\Delta t\big) \\ \boldsymbol{u}_j = \boldsymbol{u}_{j-1} + b_j \boldsymbol{k}_j \end{cases} \\
&\boldsymbol{u}^{n+1} = \boldsymbol{u}_s
\end{aligned}$$

The constants for a five-stage fourth-order Runge-Kutta scheme are

$$\begin{array}{lll}
a_1 = 0 & b_1 = \frac{1432997174477}{9575080441755} & c_1 = 0 \\
a_2 = -\frac{567301805773}{1357537059087} & b_2 = \frac{5161836677717}{13612068292357} & c_2 = \frac{1432997174477}{9575080441755} \\
a_3 = -\frac{2404267990393}{2016746695238} & b_3 = \frac{1720146321549}{2090206949498} & c_3 = \frac{2526269341429}{6820363962896} \\
a_4 = -\frac{3550918686646}{2091501179385} & b_4 = \frac{3134564353537}{4481467310338} & c_4 = \frac{2006345519317}{3224310063776} \\
a_5 = -\frac{1275806237668}{842570457699} & b_5 = \frac{2277821191437}{14882151754819} & c_5 = \frac{2802321613138}{2924317926251}
\end{array}$$

# Low-Storage and Classical RK Methods

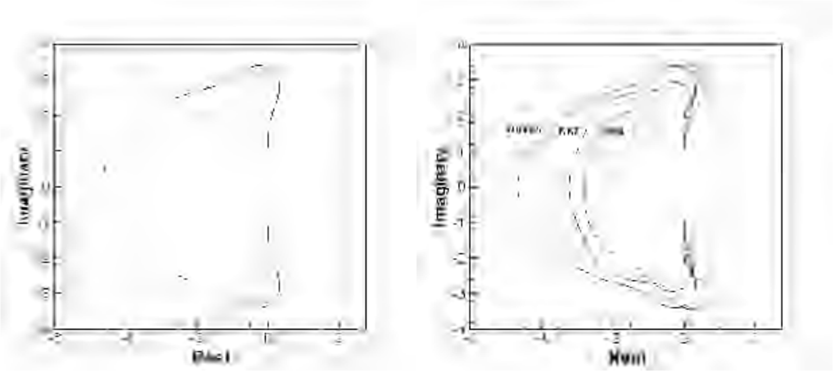



Figure: (left) Stability regions for RK4S5

- The stability region of the RK4S5 method is larger and it is suitable for advection-diffusion equation.

# Wave Equation on Spherical Surface

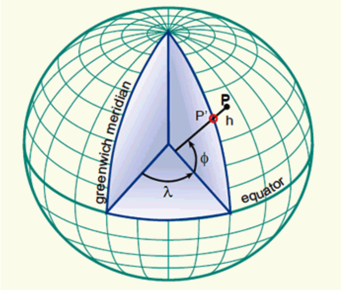
Consider the wave equation on a spherical surface:

$$\frac{\partial h}{\partial t} + \boldsymbol{V} \cdot \nabla h = 0,$$

$$h(\lambda, \phi, t = 0) = h_0(\lambda, \phi),$$

$$\boldsymbol{V} = V_\lambda(\lambda, \phi)\hat{\lambda} + V_\phi(\lambda, \phi)\hat{\phi}, \quad \nabla \cdot \boldsymbol{V} = 0$$

- $h$: depth field
- $\lambda, \phi$: longitude and latitude coordinates
- $\hat{\lambda}, \hat{\phi}$: unit vector in $\lambda$ and $\phi$ directions
- $\nabla$: surface gradient
- $\boldsymbol{V}$: wind field
- $h_0$: initial depth field

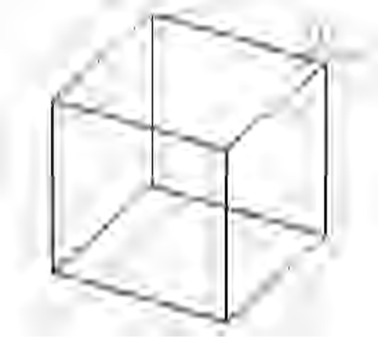


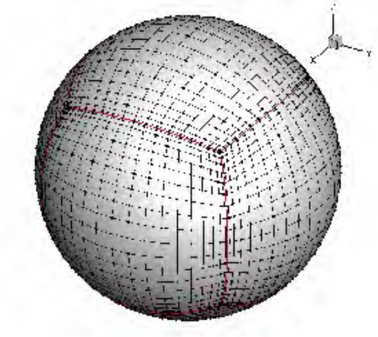

Notice that the wave equation can be expressed as

$$\frac{\partial h}{\partial t} + \boldsymbol{V} \cdot \nabla h = 0 \implies \frac{\partial h}{\partial t} + \boldsymbol{V} \cdot \nabla h + h \nabla \cdot \boldsymbol{V} = 0 \implies \frac{\partial h}{\partial t} + \nabla \cdot (\boldsymbol{V} h) = 0$$

# Cubed Sphere Mapping [1]

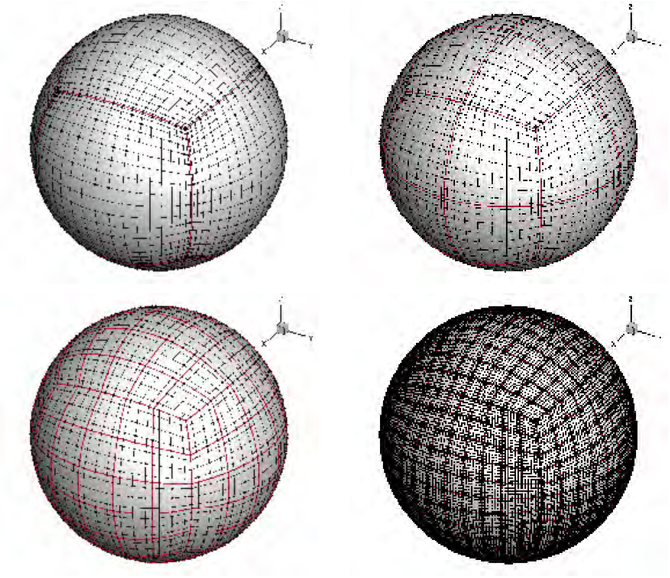
Decompose the spherical surface into 6 equal area surfaces and map them to planar surfaces of a cube.





[1] Nari, R. D., Thomas, S. J., and Loft, R. D., A discontinuous Galerkin transport scheme on the cubed sphere, Mon. Wea. Rev., 117, 130-137

# Multidomain Mesh on Spherical Surface



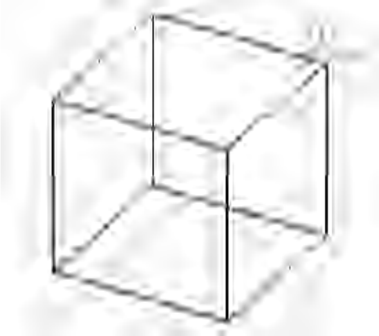
# Wave Equation on Cube Face

On each planar surface a local coordinate $(x_1, x_2)$ is set. The wave problem on each cube surface takes the form:

$$\frac{\partial Jh}{\partial t} + \frac{\partial Ju^1 h}{\partial x_1} + \frac{\partial Ju^2 h}{\partial x_2} = 0, \quad -\frac{\pi}{4} \leq x_1, x_2 \leq \frac{\pi}{4}$$
$$h(x_1, x_2, t = 0) = h_0(x_1, x_2)$$

- $h = h(x_1, x_2, t)$: mapped depth field
- $J$: coordinate mapping jacobian function
- $u^1, u^2$: contravarient components of $\boldsymbol{V}$

$$u^1 = \nabla x_1 \cdot \boldsymbol{V}$$
$$u^2 = \nabla x_2 \cdot \boldsymbol{V}$$



# Numerical Scheme

Introduce a linear mapping

$$\nu = 1,2 \quad x_\nu = x_\nu(\xi_\nu), \quad (x_1, x_2) \in D \to (\xi_1, \xi_2) \in [-1,1]^2$$

The wave problem defined on $D$ can be mapped to $I = [-1,1]^2$:

$$\frac{\partial Jh}{\partial t} + \frac{\partial Ju^1 h}{\partial x_1} + \frac{\partial Ju^2 h}{\partial x_2} = 0, (x_1, x_2) \in D \quad \frac{\partial Jh}{\partial t} + \xi_1' \frac{\partial Ju^1 h}{\partial \xi_1} + \xi_2' \frac{\partial Ju^2 h}{\partial \xi_2} = 0, (\xi_1, \xi_2) \in I$$

We propose the scheme in skew-symmetric form:

$$\begin{aligned}
\frac{d}{dt} J_{ij} h_{ij} = &- \frac{1}{2}\xi_1' \frac{\partial (Ju^1 h)}{\partial \xi_1}\Big|_{ij} - \frac{1}{2}\xi_1' (J_{ij} u^1_{ij}) \frac{\partial h}{\partial \xi_1}\Big|_{ij} - \frac{1}{2}\xi_1' \frac{\partial (Ju^1)}{\partial \xi_1}\Big|_{ij} h_{ij} \\
&- \frac{1}{2}\xi_2' \frac{\partial (Ju^2 h)}{\partial \xi_2}\Big|_{ij} - \frac{1}{2}\xi_2' (J_{ij} u^2_{ij}) \frac{\partial h}{\partial \xi_2}\Big|_{ij} - \frac{1}{2}\xi_2' \frac{\partial (Ju^2)}{\partial \xi_2}\Big|_{ij} h_{ij} \\
&- \frac{1}{2\omega_N^{\xi_1}} \tau_{Nj} J_{Nj} \delta_{Nj} \xi_1' \frac{|u^1_{Nj}| - u^1_{Nj}}{2} (h_{Nj} - h_j^+) - \frac{1}{2\omega_0^{\xi_1}} \tau_{0j} J_{0j} \delta_{0j} \xi_1' \frac{|u^1_{0j}| + u^1_{0j}}{2} (h_{0j} - h_j^-) \\
&- \frac{1}{2\omega_N^{\xi_2}} \tau_{iN} J_{iN} \delta_{Nj} \xi_2' \frac{|u^2_{iN}| - u^2_{iN}}{2} (h_{iN} - h_i^+) - \frac{1}{2\omega_0^{\xi_2}} \tau_{i0} J_{i0} \delta_{i0} \xi_2' \frac{|u^2_{i0}| + u^2_{i0}}{2} (h_{i0} - h_i^-)
\end{aligned}$$

where $h_{j/i}^{+/-}$ are the field information from other connecting elements, $\tau_{ij}$ are the point parameters, and $|u_{ij}^{1/2}| \pm u_{ij}^{1/2}|$ are called the inflow-outflow operators

# Stability

Choosing $\tau = 2$, one can show that the discrete energy rate of the scheme is bounded by the prescribed data:

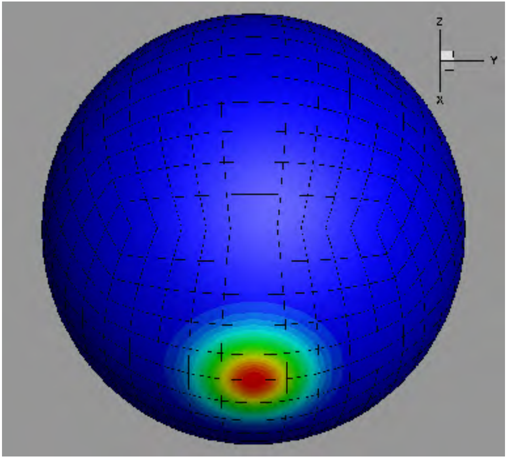
$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{d}{dt} \sum_{k=1}^{K} \sum_{j=0}^{N} \sum_{i=0}^{N} \omega_{ij} J_{ij}^{(k)} (h_{ij}^{(k)}(t))^2 \\ &= -\sum_{k=1}^{K} \sum_{j=0}^{N} \sum_{i=0}^{N} \omega_{ij} \left( (\xi_1')^{(k)} \frac{\partial (J^{(k)} u^{1\,(k)})}{\partial \xi_1} \Big|_{ij} + (\xi_2')^{(k)} \frac{\partial (J^{(k)} (u^2)^{(k)})}{\partial \xi_2} \Big|_{ij} \right) (h_{ij}^{(k)})^2 \\ &\leq \alpha \sum_{k=1}^{K} \sum_{j=0}^{N} \sum_{i=0}^{N} \omega_{ij} J_{ij}^{(k)} (h_{ij}^{(k)}) \end{aligned}$$

implying

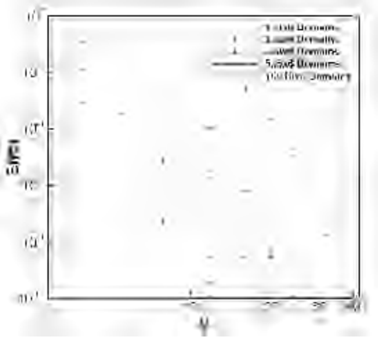
$$E(t) = \sum_{k=1}^{K} \sum_{j=0}^{N} \sum_{i=0}^{N} \omega_{ij} J_{ij}^{(k)} (h_{ij}^{(k)}(t))^2 \leq e^{\alpha t} \sum_{k=1}^{K} \sum_{j=0}^{N} \sum_{i=0}^{N} \omega_{ij} J_{ij}^{(k)} (h_{ij}^{(k)}(0))^2$$

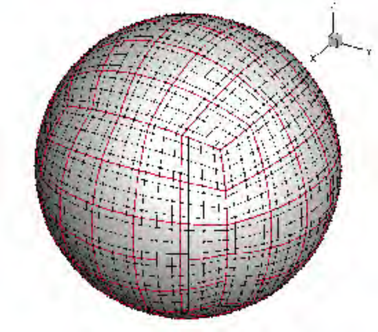
# Solid Body Rotation

## Animation of Wave Propogating on a Spherical Surface



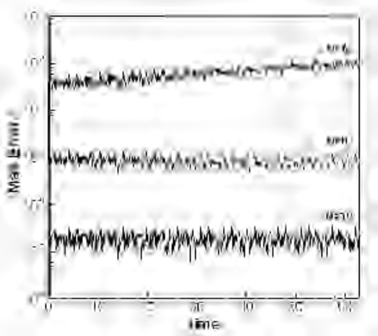
# $h$-$p$ type Convergence

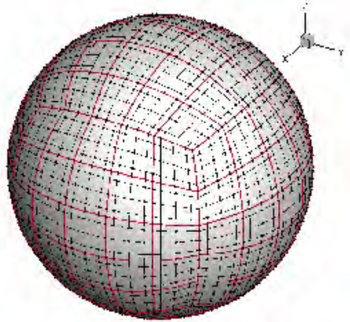




- $p$ type: Error driven down by increasing the number of elements while the degree of the approximation polynomial $N$ is fixed.
- $h$ type: Error driven down by increasing the degree of the approximation polynomial $N$ while the number of elements is fixed.

# Error History



- The error nearly grow in time (20 periods).