Numerical Partial Differential Equations:

Conservation Laws and Finite Volume Methods

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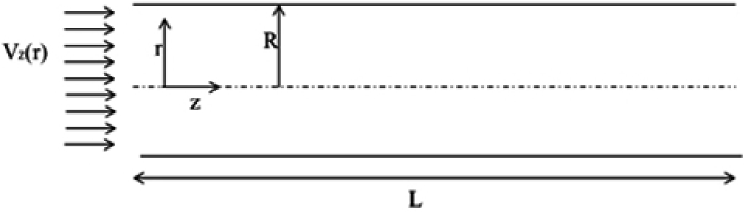
Outline

Conservation Law: An Example I



Mass conservation:

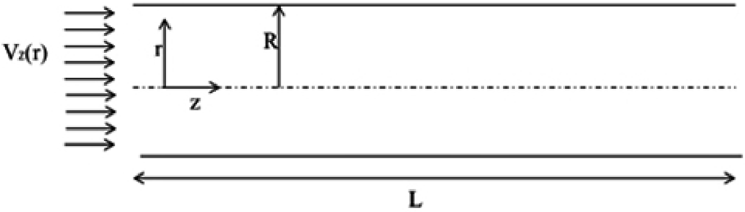
$$\begin{pmatrix}\text{Mass}\\ \text{at } t_2\end{pmatrix} - \begin{pmatrix}\text{Mass}\\ \text{at } t_1\end{pmatrix} = \begin{pmatrix}\text{Mass that}\\ \text{entered}\\ \text{from } t_1 \text{ to } t_2\end{pmatrix} - \begin{pmatrix}\text{Mass that}\\ \text{exited}\\ \text{from } t_1 \text{ to } t_2\end{pmatrix}$$

$$\int_a^b \rho(x,t_2)dx - \int_a^b \rho(x,t_1)dx = \int_{t_1}^{t_2} u(a)\rho(a,t')dt' - \int_{t_1}^{t_2} u(b)\rho(b,t')dt'$$

as $t_2 \to t_1$

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = u(a)\rho(a,t) - u(b)\rho(b,t) \tag{1}$$

Conservation Law: An Example II



Flux: $F(x,t) = u(x)\rho(x,t)$

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = u(a)\rho(a,t) - u(b)\rho(b,t) = F(a,t) - F(b,t)$$

Then

$$\int_a^b \frac{\partial \rho(x,t)}{\partial t}dx = -\int_a^b \frac{\partial F(x,t)}{\partial x}dx \quad \Longrightarrow \quad \int_a^b \left(\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x}\right) dx = 0$$

Since a and b are arbitrary constants, we have the differential mass conservation equation:

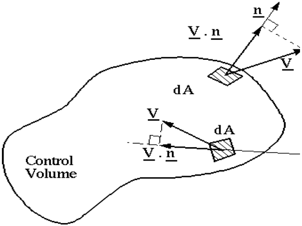
$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial F(x,t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial (u(x)\rho(x,t))}{\partial x} = 0$$

Conservation Law in Multidimensional Space

Mass conservation in integral form

$$\frac{d}{dt}\int_{\Omega}\rho(\boldsymbol{x},t)\,d\boldsymbol{x} = -\oint_{\Omega}\boldsymbol{F}(\boldsymbol{x},t)\cdot\boldsymbol{n}\,d\boldsymbol{x}$$

$$\boldsymbol{F} = \rho(x,t)\boldsymbol{V}(x)$$



If $\boldsymbol{F}$ is well defined in Ω then

$$\frac{d}{dt}\int_{\Omega}\rho(\boldsymbol{x},t)\,d\boldsymbol{x} = -\int_{\Omega}\nabla\cdot\boldsymbol{F}(\boldsymbol{x},t)\,d\boldsymbol{x} \implies \int_{\Omega}\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \nabla\cdot\boldsymbol{F}(\boldsymbol{x},t)\,d\boldsymbol{x} = 0$$

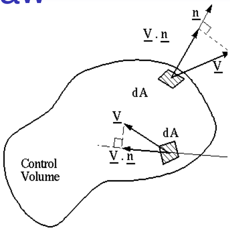
Mass conservation in differential form

$$\frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \nabla\cdot\boldsymbol{F}(\boldsymbol{x},t) = 0$$

General Conservation Law

Conserved quantities, flux functions, and source functions :

$$\boldsymbol{q}(\boldsymbol{x},t) = [q_1, q_2, \ldots, q_n]^T(\boldsymbol{x},t)$$
$$\boldsymbol{F}(x,t) = [\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n]^T(\boldsymbol{x},t)$$
$$\boldsymbol{S}(x,t) = [s_1, s_2, \ldots, s_n]^T(\boldsymbol{x},t)$$



Conservation Law

$$\frac{d}{dt}\int_\Omega \boldsymbol{q}(\boldsymbol{x},t)\,d\boldsymbol{x} = -\oint_\Omega \boldsymbol{F}\cdot\boldsymbol{n}\,d\boldsymbol{x} + \int_\Omega \boldsymbol{S}\,d\boldsymbol{x} \quad \frac{\partial \boldsymbol{q}}{\partial t} + \nabla\cdot\boldsymbol{F} = \boldsymbol{S}$$

Euler equations for fluid dynamics:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ E_t\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \boldsymbol{u}\otimes\rho\boldsymbol{u}+p\boldsymbol{I}\\ (E_t+p)\boldsymbol{u}\end{bmatrix} = \begin{bmatrix}0\\ \rho\boldsymbol{g}\\ \rho\boldsymbol{u}\boldsymbol{g}\end{bmatrix}$$

$$\frac{d}{d}\int_\Omega \begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ E_t\end{bmatrix} d\boldsymbol{x} = -\oint_\Omega \begin{bmatrix}\rho\boldsymbol{u}\\ \boldsymbol{u}\otimes\rho\boldsymbol{u}+p\boldsymbol{I}\\ (E_t+p)\boldsymbol{u}\end{bmatrix}\cdot\boldsymbol{n}\,d\boldsymbol{x} + \int_\Omega \begin{bmatrix}0\\ \rho\boldsymbol{g}\\ \rho\boldsymbol{u}\boldsymbol{g}\end{bmatrix} d\boldsymbol{x}$$

Differential or Integral Form?

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ E_t\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \boldsymbol{u}\otimes\rho\boldsymbol{u}+p\boldsymbol{I}\\ (E_t+p)\boldsymbol{u}\end{bmatrix}=\begin{bmatrix}0\\ \rho\boldsymbol{g}\\ \rho\boldsymbol{u}\boldsymbol{g}\end{bmatrix}$$

Suitable for describing fluid dynamics when flux functions are well defined ($\nabla\cdot\boldsymbol{F}$ is well defined).

$$\frac{d}{d}\int_\Omega\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ E_t\end{bmatrix}d\boldsymbol{x}=-\oint_\Omega\begin{bmatrix}\rho\boldsymbol{u}\\ \boldsymbol{u}\otimes\rho\boldsymbol{u}+p\boldsymbol{I}\\ (E_t+p)\boldsymbol{u}\end{bmatrix}\cdot\boldsymbol{n}\,d\boldsymbol{x}+\int_\Omega\begin{bmatrix}0\\ \rho\boldsymbol{g}\\ \rho\boldsymbol{u}\boldsymbol{g}\end{bmatrix}d\boldsymbol{x}$$

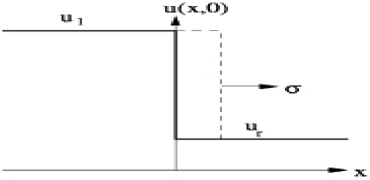
It is natural to use integral form to approximate fluid dynamics when fluid properties have abrupt changes near the shock wave surface.

Wave Problem with Discontinuity

$$\frac{\partial u(x,t)}{\partial t} + \sigma \frac{\partial u(x,t)}{\partial x} = 0,$$

$$u(x,0) = \begin{cases} u_l & x \leq 0.5 \\ u_r & x > 0.5 \end{cases}$$

$$u(0,t) = u_l$$



How do we create a system to mimic the dynamics of the problem?

- The domain is continuous. It contains infinite many points.

$$x \in [a,b], \quad t \geq 0$$

- A real number can have infinite many digits.

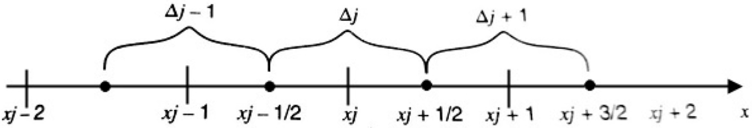
$$\pi = 3.141692654...,$$

- Computing derivatives involves a limiting process.

$$\frac{du(x)}{dx} = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

A compute is a discrete system which can not deal with terms involving infinity.

Representing the domain by a grid mesh of finite points



Grid points:

$$0 = x_0 < x_{j-\frac{1}{2}} < x_N = 1 \quad j = 1, 2, ..., N$$

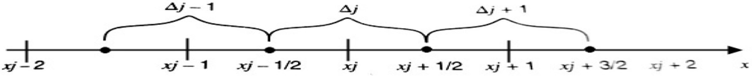
Cell and and cell width:

$$\Delta_0 = [x_0, x_{\frac{1}{2}}], \quad \Delta_j = \left[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right]_{j=1}^{N-1}, \quad \Delta_N = [x_{N-\frac{1}{2}}, x_N], \quad h_j = ||\Delta_j||$$

evaluation points and field values:

$$x_0 = 0, \quad x_j = \frac{x_{j+\frac{1}{2}} + x_{j-\frac{1}{2}}}{2}, \quad x_N = 1,$$
$$v_j(t) \approx u(x_j, t), \quad f_j(t) = f(v_j, t) \approx f(u(x_j, t))$$

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0, \quad \frac{d}{dt}\int_a^b u(x,t)\,dx = f(u(a,t)) - f(u(b,t))$$



finite difference scheme:

$$\frac{dv_j(t)}{dt} = -\frac{f(v_{j+1}(t)) - f(v_{j-1}(t))}{2h}, \quad v_j(t) \approx u(x_j, t),$$

finite volume scheme 1:

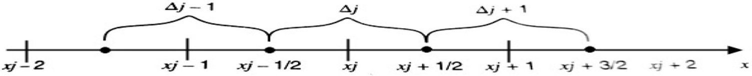
$$\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} u(x,t)\,dx \approx \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)\,dx = Q_i(t), \quad \text{cell averaged value of } u$$

$$f(u(x_{i+\frac{1}{2}})) \approx \frac{f(v_j) + f(v_{j+1})}{2}, \quad f(u(x_{i-\frac{1}{2}})) \approx \frac{f(v_{j-1}) + f(v_j)}{2},$$

$$\frac{d}{dt}\left(\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)\,dx\right) = \frac{f(v_{j-1}) + f(v_j)}{2h} - \frac{f(v_j) + f(v_{j+1})}{2h} = -\frac{f(v_{j+1}) - f(v_{j-1})}{2h}$$

$$\frac{dQ_i(t)}{dt} = -\frac{f(v_{j+1}) - f(v_{j-1})}{2h}, \quad \text{Note } Q_i(t) = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)\,dx = v_j(t)$$

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0, \quad \frac{d}{dt}\int_a^b u(x,t)\,dx = f(u(a,t)) - f(u(b,t))$$



finite difference scheme:

$$\frac{dv_j(t)}{dt} = -\frac{f(v_{j+1}(t)) - f(v_{j-1}(t))}{2h}, \quad v_j(t) \approx u(x_j,t),$$

finite volume scheme 2:

$$\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} u(x,t)\,dx \approx \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)\,dx = Q_i(t) = v_j(t), \quad \text{cell averaged value of } u$$

$$f(u(x_{i+\frac{1}{2}})) \approx f\left(\frac{v_j + v_{j+1}}{2}\right), \quad f(u(x_{i-\frac{1}{2}})) \approx f\left(\frac{v_{j-1} + v_j}{2}\right),$$

$$\frac{d}{dt}\left(\frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)\,dx\right) = -\frac{1}{h}\left(f\left(\frac{v_j + v_{j+1}}{2}\right) - f\left(\frac{v_{j-1} + v_j}{2}\right)\right)$$

$$\frac{dQ_i(t)}{dt} = -\frac{f(\bar{v}_{j+1/2}) - f(\bar{v}_{j-1/2})}{h}, \quad \bar{v}_{j+1/2} = \frac{v_j + v_{j+1}}{2}$$

finite volume scheme 1:

$$\frac{dQ_i(t)}{dt} = -\frac{f(v_{j+1}) - f(v_{j-1})}{2h},$$

$$Q_i(t) = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v_j(t)\,dx = v_j(t)$$

finite volume scheme 2:

$$\frac{dQ_i(t)}{dt} = -\frac{f(\bar{v}_{j+1/2}) - f(\bar{v}_{j-1/2})}{h},$$

$$\bar{v}_{j+1/2} = \frac{v_j + v_{j+1}}{2}$$

General form of a finite volume scheme:

$$\frac{dQ_i(t)}{dt} = -\frac{F_{i+1/2} - F_{i-1/2}}{h}, \quad Q_i = \frac{1}{h}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v_j(t)dx = v_j(t)$$

$$F_{i+1/2} = \mathcal{F}(Q_{i-r}, \ldots, Q_{i-1}, Q_i, Q_{i+1}, \ldots, Q_{i+s})$$

Numerical flux function for FV scheme 1:

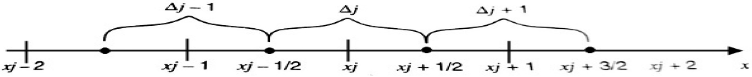
$$F_{i+1/2} = \frac{f(v_j) + f(v_{j+1})}{2} = \frac{1}{2}(f(Q_j) + f(Q_{j+1})) = \mathcal{F}(Q_j, Q_{j+1})$$

$$F_{i+1/2} - F_{i-1/2} = \mathcal{F}(Q_j, Q_{j+1}) - \mathcal{F}(Q_{j-1}, Q_j) = \frac{f(v_j) + f(v_{j+1})}{2} - \frac{f(v_{j-1}) + f(v_j)}{2}$$

Numerical flux function for FV scheme 2:

$$F_{i+1/2} = f(\frac{v_j + v_{j+1}}{2}) = f(\frac{Q_j + Q_{j+1}}{2}) = \mathcal{F}(Q_j, Q_{j+1})$$

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0, \quad \frac{d}{dt}\int_a^b u(x,t)\,dx = f(u(a,t)) - f(u(b,t))$$



Finite volume scheme:

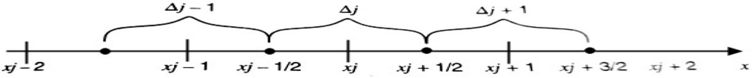
$$\frac{dQ_i(t)}{dt} = -\frac{F_{i+1/2} - F_{i-1/2}}{h_i}, \quad Q_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} v_j(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_{i-r}, ..., Q_{i-1}, Q_i, Q_{i+1}, ..., Q_{i+s})$$

- Allow non-uniform cell size.
- $\mathcal{F}$ is called numerical flux function, and $Q_{i-n}, ..., Q_{i+m}$ are neighboring cell averaged values.
- Allow constructing $\mathcal{F}$ using unsymmetrical stencil.
- Conservation property of the scheme is satisfied automatically,

$$\frac{d}{dt}\sum_{j=1}^{J} h_j Q_j = \sum_{j=1}^{J} h_j \frac{dQ_j}{dt} = -\sum_{j=1}^{J}(F_{j+1/2} - F_{j-1/2}) = F_{-1/2} + F_{J+1/2}$$

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0, \quad \frac{d}{dt}\int_a^b u(x,t)\,dx = f(u(a,t)) - f(u(b,t))$$



Finite volume scheme:

$$\frac{dQ_i(t)}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i} = 0, \quad Q_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_{i-r}, \ldots, Q_{i-1}, Q_i, Q_{i+1}, \ldots, Q_{i+s})$$

Convergence: $v_j(t) \to u(x_j, t)$ as $h_j \to 0$

1. The scheme is *consistent* with the differential equation, meaning that the scheme approximate the differental equation well locally.
2. The scheme is *stable* in some appropriate sense, meaning that at any given terminal time T the numerical solution $v_j(T)$ is bounded by the data (initial and boundary conditions), independent of the grid size.

Finite volume scheme:

$$\frac{dQ_i(t)}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i} = 0,$$

$$Q_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_{i-r}, ..., Q_{i+s})$$

Equations:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0,$$

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = f(u(a,t)) - f(u(b,t))$$

Truncation error of the scheme:

$$R_i(t) = \frac{d\bar{Q}_i}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i},$$

$$\bar{Q}_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} u(x_j,t)dx = u(x_j,t),$$

$$F_{i+1/2} = \mathcal{F}(\bar{Q}_{i-r}, ..., \bar{Q}_{i+s})$$

Consistency:

$R_i \to 0$ as $h_i \to 0,$ or $\mathcal{F}(\bar{q}, ..., \bar{q}) \to f(\bar{q})$

Example:

$$\frac{dQ_i(t)}{dt} = -\frac{f(Q_{i+1}) - f(Q_{i-1})}{2h}$$

$$\begin{aligned}R_i(t) &= \frac{\partial u(x_i,t)}{\partial t} + \frac{f(u(x_{i+1},t)) - f(u(x_{i-1},t))}{2h}\\ &= \frac{\partial u(x_i,t)}{\partial t} + \frac{\partial f(u(x_i,t))}{\partial x} + \mathcal{O}(h^2)\\ &= \mathcal{O}(h^2) \to 0 \text{ as } h \to 0\end{aligned}$$

$$\mathcal{F}(\bar{q}, \bar{q}) = \frac{f(\bar{q}) + f(\bar{q})}{2} = f(\bar{q})$$

Finite volume scheme:

$$\frac{dQ_i(t)}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i} = 0,$$

$$Q_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_{i-r}, ..., Q_{i+s})$$

Equations:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u(x,t))}{\partial x} = 0,$$

$$\frac{d}{dt}\int_a^b u(x,t)\,dx = f(u(a,t)) - f(u(b,t))$$

Consistency and truncation error of the scheme:

$$R_i(t) = \frac{d\bar{Q}_i}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i}, \quad \bar{Q}_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} u(x_i,t)dx = u(x_i,t),$$

$$F_{i+1/2} = \mathcal{F}(\bar{Q}_{i-r}, ..., \bar{Q}_{i+s}), \quad R_i \to 0 \text{ as } h_i \to 0, \text{ or } \mathcal{F}(\bar{q}, ..., \bar{q}) \to f(\bar{q})$$

Example: $\frac{dQ_i(t)}{dt} = -\frac{f(Q_{i+1}) - f(Q_{i-1})}{2h}$

$$R_i(t) = \frac{\partial u(x_i,t)}{\partial t} + \frac{f(u(x_{i+1},t)) - f(u(x_{i-1},t))}{2h} = \frac{\partial u(x_i,t)}{\partial t} + \frac{\partial f(u(x_i,t))}{\partial x} + \mathcal{O}(h^2)$$

$$= \mathcal{O}(h^2) \to 0 \text{ as } h \to 0$$

$$\mathcal{F}(\bar{q},\bar{q}) = \frac{f(\bar{q}) + f(\bar{q})}{2} = f(\bar{q})$$

Stability Analysis

Stability analysis for schemes dependents on the type of the partial differential equations.

- CFL condition
- Lax-Richtmyer stability for liner methods
- von-Neumann analysis (suitable for constant coefficient linear problems + periodic boundary conditions)
- GKS theory or normal mode analysis (extension of von-Neumann analysis with non-periodic boundary conditions)
- Energy method: 2-norm, 1-norm, and ∞-norm
- Total-Variation (TV) stability analysis for nonlinear methods

L_2 Stability Method for Variable Coefficient Problem

Let $f(u) = a(x)u(x,t)$, $a(x) > 0$. Consider the problem:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, && x \in [0,1], \quad t \geq 0, \\
&u(x,0) = u_0(x), && x \in [0,1] \\
&u(0,t) = g(t), && t \geq 0
\end{aligned}
$$

FVM scheme:

$$
\begin{aligned}
&\frac{dv_0}{dt} = -\frac{f_1 - f_0}{2h_0} - \frac{\tau_-}{h_0}(f_0 - f_-), && \text{left cell}, \tau_- = 1 \\
&\frac{dv_j}{dt} = -\frac{f_{j+1} - f_{j-1}}{2h_j}, && j = 1, ..., N-1 \\
&\frac{dv_N}{dt} = -\frac{f_N - f_{N-1}}{2h_N} && \text{right cell} \\
&v_j(0) = u_0(x_j) && j = 0, 1, ..., N
\end{aligned}
$$

L_2 Stability: $a_j v_j = f_j$, Find $\sum_{j=0}^{N} a_j h_j v_j^2$

$$\frac{dv_0}{dt} = -\frac{f_1 - f_0}{2h_0} - \frac{1}{h_0}(f_0 - f_-), \quad \underbrace{\frac{dv_j}{dt} = -\frac{f_{j+1} - f_{j-1}}{2h_j},}_{j=1,2,\ldots,N-1} \quad \frac{dv_N}{dt} = -\frac{f_N - f_{N-1}}{2h_N}$$

$$2v_0 a_0 h_0 \frac{dv_0}{dt} = -f_0(f_1 - f_0) - 2f_0(f_0 - f_-) = -f_0 f_1 - f_0^2 + 2f_0 f_-$$

$$\sum_{j=1}^{N-1} 2v_j a_j h_j \frac{dv_j}{dt} = -\sum_{j-1}^{N-1} f_j(f_{j+1} - f_{j-1}) = f_1 f_0 - f_{N-1} f_N,$$

$$2v_N a_N h_N \frac{dv_N}{dt} = f_N f_{N-1} - f_N^2$$

$$\frac{d}{dt}\sum_{j=0}^{N} a_j h_j v_j^2(t) = -f_0^2 + 2f_0 f_- - f_N^2 = -f_N^2 - (f_0 - f_-(t))^2 + f_-^2(t) \le f_-^2(t)$$

$$\Longrightarrow \sum_{j=0}^{N} a_j h_j v_j^2(t) \le \sum_{j=0}^{N} a_j h_j v_j^2(0) + \int_0^t f_-^2(t')dt' = M \Longrightarrow \sum_{j=0}^{N} h_j v_j^2(t) < cM$$

Model Equations

Consider the partial differential equation of the form:

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon(x)\frac{\partial^2 u(x,t)}{\partial x^2}$$

wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(a(x,t)u(x,t)\right) = 0$$

advection-diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(a(x,t)u(x,t)\right) = \varepsilon(x)\frac{\partial^2 u}{\partial x^2}$$

heat equation

$$\frac{\partial u}{\partial t} = \varepsilon(x)\frac{\partial^2 u}{\partial x^2}$$

viscous Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2(x,t)}{2}\right) = \varepsilon(x)\frac{\partial^2 u}{\partial x^2}$$

The equation can be also written as

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(f(u) - \varepsilon\frac{\partial u}{\partial x}\right) = 0, \quad F(u) = f(u) - \varepsilon\frac{\partial u}{\partial x} : \text{ viscous flux}$$

which is in the form of a conservation law.

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f(u) = au(x,t), \quad a > 0$$

$$\frac{dQ_i(t)}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i} = 0, \quad Q_i = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_i, Q_{i+1}) = \frac{1}{2}\left(f(Q_i) + f(Q_{i+1})\right)$$

Semi-discrete scheme

$$\frac{dQ_i(t)}{dt} = -a\left(\frac{Q_{i+1} - Q_{i-1}}{2h_i}\right)$$

Euler scheme: ($ak/2/h_i^2 < 1$)

$$\frac{Q_i^{n+1} - Q_i^{n+1}}{k} = -a\frac{Q_{i+1}^n - Q_{i-1}^n}{2h_i}, \quad Q_i^{n+1} = Q_i^n + \frac{ak}{2h_i}\left(Q_{i+1}^n - Q_{i-1}^n\right)$$

Lax-Friedrichs scheme:

$$Q_i^{n+1} = \frac{Q_{i-1}^n + Q_{i+1}^n}{2} + \frac{ak}{2h_i}\left(Q_{i+1}^n - Q_{i-1}^n\right)$$

Leap frog: ($ak/h_i < 1$)

$$\frac{Q_i^{n+1} - Q_i^{n-1}}{2k} = -a\frac{Q_{i+1}^n - Q_{i-1}^n}{2h_i}, \quad Q_i^{n+1} = Q_i^{n-1} + \frac{ak}{h_i}\left(Q_{i+1}^n - Q_{i-1}^n\right)$$

$$\frac{\partial u}{\partial t} = \varepsilon(x)\frac{\partial^2 u}{\partial x^2}, \quad \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad f(u) = -\varepsilon(x)\frac{\partial u}{\partial x}$$

$$\frac{dQ_i}{dt} = -\frac{1}{h_i}\left(F_{i+1/2} - F_{i-1/2}\right), \quad Q_i(t) = \frac{1}{h_i}\int_{x_{i-\frac{1}{2}}}^{x_i+\frac{1}{2}} v_j(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_i, Q_{i+1}) = -\varepsilon(x_{i+1/2})\left(\frac{Q_{i+1} - Q_i}{h_i}\right)$$

$$\frac{dQ_i}{dt} = \frac{1}{h_i^2}\left(\varepsilon(x_{i+1/2})\left(Q_{i+1} - Q_i\right) - \varepsilon(x_{i-1})\left(Q_i - Q_{i-1/2}\right)\right)$$

If $\varepsilon(x) = \varepsilon$ then

$$\frac{dQ_i}{dt} = \varepsilon\left(\frac{Q_{i+1} - 2Q_i + Q_{i-1}}{h_i^2}\right)$$

Crank-Nicolson method

$$\frac{Q_i^{n+1} - Q_i^n}{k} = \varepsilon\frac{Q_{i+1}^{n+1/2} - 2Q_i^{n+1/2} + Q_{i-1}^{n+1/2}}{h_i^2}, \quad Q_j^{n+1/2} = \frac{Q_j^{n+1} + Q_j^n}{2}$$