Numerical Partial Differential Equations:

Conservation Laws and Finite Volume Methods (II)

Chun-Hao Teng

Department of Applied Mathematics
National Chung Hsing University, Taichung 701, Taiwan

Outline

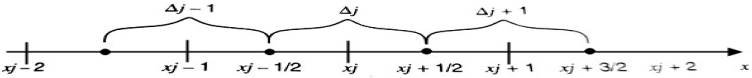
Outline

1. Basic Concepts of FVM
   - Monotonicity/TVD/L_1-Contracting/monotone methods
2. van-Leer Type Methods
   - van-Leer method with different mismatch values
3. 2/3D Problem and Splitting Methods
   - Splitting methods

Conservation Law and FVM

Consider $u = u(x,t)$ satisfying the 2π periodic problem:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad u(x,0) = u_0(x), \qquad x \in [0, 2\pi), t > 0$$



Grid points, cells, cell centers and evaluated fields

$$x_j = h \cdot j, \quad h = \frac{2\pi}{N+1}, \quad j = 0, 1, ...N.$$

$$\Delta_j = [x_{j-1/2}, x_{j+1/2}]_{j=0}^{N}, \quad v_j(t) \approx u(x_j, t)$$

General Form of FV scheme:

$$\frac{dQ_i(t)}{dt} = -\frac{F_{i+1/2} - F_{i-1/2}}{h}, \quad Q_i = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} v_i(t)dx$$

$$F_{i+1/2} = \mathcal{F}(Q_{i-r}, ..., Q_i, ..., Q_{i+s})$$

Model Examples

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \qquad u(x,0) = u_0(x), \qquad x \in [0, 2\pi), t > 0$$

$$\frac{dQ_i(t)}{dt} + \frac{F_{i+1/2} - F_{i-1/2}}{h} = 0, \quad Q_i = \frac{1}{h}\int_{x_{i-1/2}}^{x_{i+1/2}} v_i(t)dx$$

$$F_{i-1/2} = \mathcal{F}(Q_{i-1}, Q_i) = \frac{1}{2}\left(f(Q_{i-1}) + f(Q_i)\right)$$

Semi-discrete scheme

$$\frac{dQ_i(t)}{dt} = -\left(\frac{f(Q_{i+1}) - f(Q_{i-1})}{2h}\right)$$

Upwind scheme: $(f'(u) > 0, \max f'(u))k/h \leq 1, ' = d/du)$

$$\frac{Q_i^{n+1} - Q_i^n}{k} = -\frac{f(Q_i^n) - f(Q_{i-1}^n)}{h},$$

$$Q_i^{n+1} = Q_i^n + \frac{k}{h}\left(f(Q_i^n) - f(Q_{i-1}^n)\right) = \mathcal{H}(Q_i^n; i)$$

Lax-Friedrichs scheme:

$$Q_i^{n+1} = \frac{Q_{i-1}^n + Q_{i+1}^n}{2} + \frac{k}{2h}\left(f(Q_{i+1}^n) - f(Q_{i-1}^n)\right) = \mathcal{H}(Q^n; i)$$

Monotonicity Preserving Method

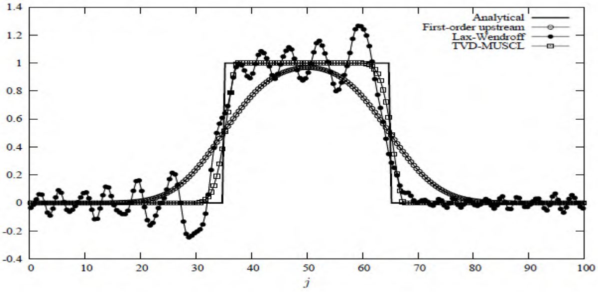
Monotonicity preserving: Consider the method

$$Q_j^{n+1} = \mathcal{H}(Q^n; j)$$

If for all j

$$Q_j^n \geq Q_{j+1}^n \implies Q_j^{n+1} \geq Q_{j+1}^{n+1}$$

If the initial data Q_j^0 is monotone (either non-increasing or non-decreasing) as a function of j, then the solution Q_j^n should have the same property for all n.

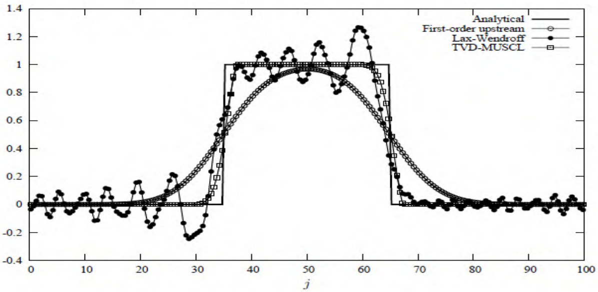


This means in particular that oscillations can not arise near an isolated propagating discontinity.

Total Variation Diminishing (TVD)

A numerical method is total variation non-increasing (or, less accurate but nonetheless more often used total variation diminishing or TVD), when

$$TV(U^{n+1}) \leq TV(U^n), \quad TV(U) = \sum_{j=j_{min}}^{j_{max}} |U_{j+1} - U_j|.$$



A TVD scheme is a monotonicity preserving method.

Upwind scheme: $Q_i^{n+1} = Q_i^n - \frac{k}{h}\left(f(Q_i^n) - f(Q_{i-1}^n)\right), f'(u) > 0,$ $(\max f'(u))(k/h) \leq 1, ' = d/du$

We have

$$Q_{i+1}^{n+1} = Q_{i+1}^n - \frac{k}{h}\left(f(Q_{i+1}^n) - f(Q_i^n)\right), \quad \text{and} \quad f'(\theta_{i+1}^n) = \frac{f(Q_{i+1}^n) - f(Q_i^n)}{Q_{i+1}^n - Q_i^n}$$

where θ_{i+1}^n is between Q_{i+1}^n and Q_i^n. Then

$$\begin{aligned} Q_{i+1}^{n+1} - Q_i^{n+1} &= Q_{i+1}^n - Q_i^n - \frac{k}{h}\left(f(Q_{i+1}^n) - f(Q_i^n)\right) + \frac{k}{h}\left(f(Q_i^n) - f(Q_{i-1}^n)\right) \\ &= Q_{i+1}^n - Q_i^n - \frac{k}{h}f'(\theta_{i+1}^n)\left(Q_{i+1}^n - Q_i^n\right) + \frac{k}{h}f'(\theta_i^n)\left(Q_i^n - Q_{i-1}^n\right) \\ &= \left(1 - \frac{k}{h}f'(\theta_{i+1}^n)\right)\left(Q_{i+1}^n - Q_i^n\right) + \frac{k}{h}f'(\theta_i^n)\left(Q_i^n - Q_{i-1}^n\right) \end{aligned}$$

Hence

$$\begin{aligned} |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq \left(1 - \frac{k}{h}f'(\theta_{i+1}^n)\right)\left|Q_{i+1}^n - Q_i^n\right| + \frac{k}{h}f'(\theta_i^n)\left|Q_i^n - Q_{i-1}^n\right| \\ &= \left|Q_{i+1}^n - Q_i^n\right| - \frac{k}{h}f'(\theta_{i+1}^n)\left|Q_{i+1}^n - Q_i^n\right| + \frac{k}{h}f'(\theta_i^n)\left|Q_i^n - Q_{i-1}^n\right| \end{aligned}$$

leading to

$$\begin{aligned} \sum_i |Q_{i+1}^{n+1} - Q_i^{n+1}| &\leq \sum_i \left|Q_{i+1}^n - Q_i^n\right| - \frac{k}{h}\sum_i f'(\theta_{i+1}^n)\left|Q_{i+1}^n - Q_i^n\right| + \frac{k}{h}\sum_i f'(\theta_i^n)\left|Q_i^n - Q_{i-1}^n\right| \\ &= \sum_i \left|Q_{i+1}^n - Q_i^n\right| \leq \sum_i \left|Q_{i+1}^{n-1} - Q_i^{n-1}\right| \leq \cdots \leq \sum_i \left|Q_{i+1}^0 - Q_i^0\right| \end{aligned}$$

L_1-Contracting Method $\quad Q_j^{n+1} = \mathcal{H}(Q^n; j)$

We say a numerical method is L_1 contracting, if U_j^n, V_j^n, $U_j^{n+1} = \mathcal{H}(U^n; j)$ and $V_j^{n+1} = \mathcal{H}(V^n; j)$ satisfy

$$||U^{n+1} - V^{n+1}||_1 \leq ||U^n - V^n||_1, \quad ||Q^n||_1 = \sum_{j=0}^{N} |Q_j^n| h, \quad \text{(discrete } L_1\text{-norm)}$$

Upwind scheme

$$Q_i^{n+1} = Q_i^n - \frac{k}{h}\left(f(Q_i^n) - f(Q_{i-1}^n)\right), \qquad \begin{aligned} U_i^{n+1} &= U_i^n - \tfrac{k}{h}\left(f(U_i^n) - f(U_{i-1}^n)\right) \\ V_i^{n+1} &= V_i^n - \tfrac{k}{h}\left(f(V_i^n) - f(V_{i-1}^n)\right) \end{aligned}$$

Let $W_i^n = U_i^n - V_i^n$. Then

$$\begin{aligned} W_i^{n+1} &= W_i^n - \frac{k}{h}\left(f(U_i^n) - f(V_i^n)\right) + \frac{k}{h}\left(f(U_{i-1}^n) - f(V_{i-1}^n)\right) \qquad \left(f'(\theta_i) = \frac{f(U_i) - f(V_i)}{U_i - V_i}\right) \\ &= W_i^n - \frac{kf'(\theta_i^n)}{h}(U_i^n - V_i^n) + \frac{kf'(\theta_{i-1}^n)}{h}\left(U_{i-1}^n - V_{i-1}^n\right) \\ &= (1-\alpha_i)W_j^n + \alpha_{i-1}^n W_{i-1}^n, \quad 0 \leq \alpha_i = \frac{kf'(\theta_i^n)}{h} \leq 1 \end{aligned}$$

We have

$$|W_i^{n+1}| \leq (1-\alpha_i)|W_i^n| + \alpha_{i-1}^n |W_{i-1}^n|$$

$$\Longrightarrow \quad ||W^{n+1}||_1 \leq ||W^n||_1 - \sum_i \alpha_i^n |W_i^n| h + \sum_i \alpha_{i-1}^n |W_{i-1}^n| h = ||W^n||_1$$

Monotone Method

The numerical method

$$U_j^{n+1} = \mathcal{H}(U^n; j)$$

is called a **monotone method** if the following property holds:

$$V_j^n \geq U_j^n \quad \forall j \quad \Longrightarrow \quad V_j^{n+1} \geq U_j^{n+1} \quad \forall j$$

To prove that a method is monotone, it is suffices to check that

$$\frac{\partial \mathcal{H}(U^n; j)}{\partial U_i^n} \geq 0 \quad \text{for all } i, j, U^n$$

Lax-Friedrichs scheme:

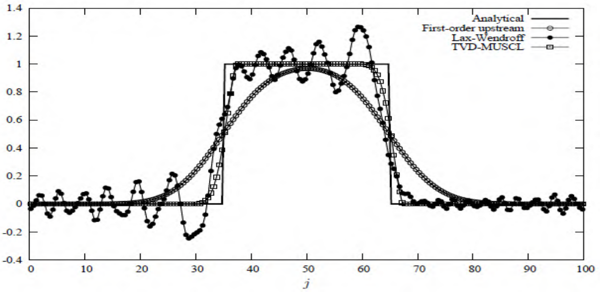
$$Q_i^{n+1} = \frac{Q_{i-1}^n + Q_{i+1}^n}{2} + \frac{k}{2h}\left(f(Q_{i+1}^n) - f(Q_{i-1}^n)\right) = \mathcal{H}(Q^n; i)$$

$$\frac{\partial \mathcal{H}(Q^n; i)}{\partial Q_j^n} = \begin{cases} \frac{1}{2}(1 + \frac{k}{h} f'(Q_{i-1}^n)) & \text{if } j = i-1 \\ \frac{1}{2}(1 - \frac{k}{h} f'(Q_{i+1}^n)) & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Monotone methods are first order accurate.

Monotonicity-Preserving/TVD/L_1-Contracting/Monotone Methods

monotone $\Longrightarrow$ L_1-contracting $\Longrightarrow$ TVD $\Longrightarrow$ monotonicity-preserving



A method should have enough disspitation to smear the oscilations without harming the accuracy.

Outline

1. Basic Concepts of FVM
   - Monotonicity/TVD/L_1-Contracting/monotone methods
2. van-Leer Type Methods
   - van-Leer method with different mismatch values
3. 2/3D Problem and Splitting Methods
   - Splitting methods

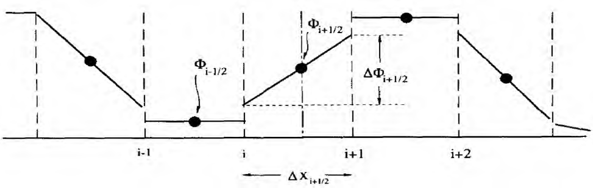
References

- A class of the van Leer-type transport schemes and its application to the moisture transport in a general circulation model,
  S.-J. Lin, W. C. Chao, and G. K. Walker, Mon. Wea. Rev. (1994), 122, 1575-1593.
- Multidimensional flux-form semi-Lagrangian transport schemes,
  S.-J. Lin and R. Rood, Mon. Wea. Rev. (1996), 124, 2046-2070.
- An explicit flux-form semi-Lagrangian shallow-water model on the sphere,
  S.-J. Lin and R. Rood, Q. J. R. Meteorol. Soc. (1997), 123, 2477-2498.
- A "vertically Lagrangian" finite-volume dynamical core for global models,
  S.-J. Lin, Mon. Wea. Rev. (2004), 132, 2293-2307.
- Finite-volume transport on various cubed-sphere grids,
  W. M. Putman and S. J. Lin, J. Comput. Phys. (2007), 227, 55-78.
- A two-way nested global-regional dynamical core on the cubed-sphere grid
  L. M. Harris, S.-J. Lin, Mon. Wea. Rev. (2013), 141, 283-306.

Van-Leer type method for $\rho_t(x,t) + f_x(x,t) = 0, f(x,t) = u(x,t)\rho(x,t)$

Transport equatation in integral form:

$$\frac{d}{dt}\left(\frac{1}{b-a}\int_b^a \rho(x,t)dx\right)dt = \frac{-1}{(b-a)}\left(f(b,t) - f(a,t)\right) \tag{1}$$



FV scheme:

$$\frac{1}{\Delta x_{i+1/2}}\int_{x_i}^{x_{i+1}} \rho(x,t_n)dx \approx \frac{1}{\Delta x_{i+1/2}}\int_{x_i}^{x_{i+1}} \Phi^n_{i+1/2} + \frac{\Delta\Phi_{i+1/2}}{2}(x - x_{i+1/2})dx = \Phi^n_{i+1/2}$$

$$\Phi^{n+1}_{i+1/2} = \Phi^n_{i+1/2} - \frac{\Delta t}{\Delta x_{i+1/2}}(F_{i+1} - F_i)$$

if $U_i \geq 0, \quad F_i = U_i\left(\Phi^n_{i-1/2} + \frac{\Delta\Phi_{i-1/2}}{2}(1 - C_i^-)\right), \quad C_i^- = \frac{U_i\Delta t}{\Delta x_{i-1/2}}$

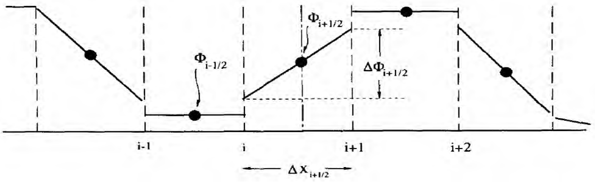
if $U_i \geq 0, \quad F_i = U_i\left(\Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C_i^+)\right), \quad C_i^+ = \frac{U_i\Delta t}{\Delta x_{i+1/2}}$

Note that mismatch $\Delta\Phi_i \geq 0$.

Van-Leer type method for $\rho_t(x,t) + f_x(x,t) = 0, f(x,t) = u(x,t)\rho(x,t)$

Transport equatation in integral form:

$$\frac{d}{dt}\left(\frac{1}{b-a}\int_b^a \rho(x,t)dx\right) dt = \frac{-1}{(b-a)}\left(f(b,t) - f(a,t)\right)$$



FV scheme: if mismatch (slope) $\Delta\Phi_{i+1/2} = 0$

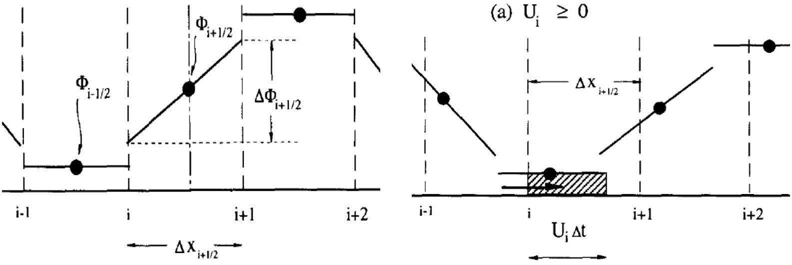
$$\Phi^{n+1}_{i+1/2} = \Phi^n_{i+1/2} - \frac{\Delta t}{\Delta x_{i+1/2}}(F_{i+1} - F_i)$$

$$\text{if } U_i \geq 0, \quad F_i = U_i\left(\Phi^n_{i-1/2} + \frac{\Delta\Phi_{i-1/2}}{2}(1 - C^-_i)\right) = U_i\Phi^n_{i-1/2}, \quad C^-_i = \frac{U_i\Delta t}{\Delta x_{i-1/2}}$$

$$\text{if } U_i \geq 0, \quad F_i = U_i\left(\Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C^+_i)\right) = U_i\Phi^n_{i+1/2}, \quad C^+_i = \frac{U_i\Delta t}{\Delta x_{i+1/2}}$$

Pure upwind method and 1st-order accurate

Flux function

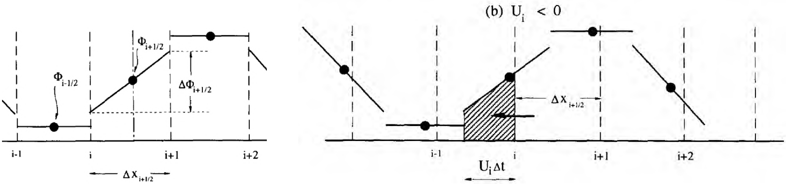


$$\Phi_{i+1/2}^{n+1} = \Phi_{i+1/2}^{n} - \frac{\Delta t}{\Delta x_{i+1/2}}(F_{i+1} - F_i)$$

$$\text{if } U_i \geq 0, \quad F_i = U_i\left(\Phi_{i-1/2}^{n} + \frac{\Delta\Phi_{i-1/2}}{2}(1 - C_i^-)\right), \quad C_i^- = \frac{U_i \Delta t}{\Delta x_{i-1/2}}$$

$$\text{if } U_i < 0, \quad F_i = U_i\left(\Phi_{i+1/2}^{n} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C_i^+)\right), \quad C_i^+ = \frac{U_i \Delta t}{\Delta x_{i+1/2}}$$

Flux function II



$$\Phi^{n+1}_{i+1/2} = \Phi^n_{i+1/2} - \frac{\Delta t}{\Delta x_{i+1/2}}(F_{i+1} - F_i)$$

$$\text{if } U_i \geq 0, \quad F_i = U_i\left(\Phi^n_{i-1/2} + \frac{\Delta\Phi_{i-1/2}}{2}(1 - C_i^-)\right), \quad C_i^- = \frac{U_i \Delta t}{\Delta x_{i-1/2}}$$

$$\text{if } U_i < 0, \quad F_i = U_i\left(\Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C_i^+)\right), \quad C_i^+ = \frac{U_i \Delta t}{\Delta x_{i+1/2}}$$

Note that

$$C_i^+ \to 0 \implies \Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C_i^+) \to \Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2} = \Phi^n_i$$

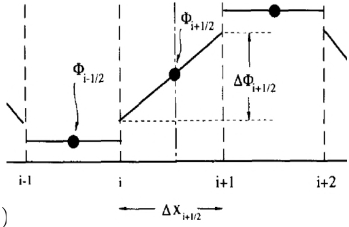
$$C_i^+ = -\frac{1}{2} \implies \Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C_i^+) = \Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{4}$$

$$C_i^+ \to -1 \implies \Phi^n_{i+1/2} - \frac{\Delta\Phi_{i+1/2}}{2}(1 + C_i^+) \to \Phi^n_{i+1/2}$$

How to Choose $\Phi_{i+1/2}$?

- algebraic mean:

$$\Delta\Phi_{i+1/2} = [\Delta\Phi_{i+1/2}]_{ave} = \frac{\delta\Phi_i + \delta\Phi_{i+1}}{2},$$
$$\delta\Phi_i = \Phi_{i+1/2} - \Phi_{i-1/2}$$



- local lower bound required: $\Phi_{i+\frac{1}{2}} \geq \Phi^{min}_{i+\frac{1}{2}} \geq 0$

$$[\Delta\Phi_{i+\frac{1}{2}}]_{posd} = \text{sign}([\Delta\Phi_{i+\frac{1}{2}}]_{ave}) \times \min(|[\Delta\Phi_{i+\frac{1}{2}}]_{ave}|, 2|\Phi_{i+\frac{1}{2}} - \Phi^{min}_{i+\frac{1}{2}}|)$$

- local lower/upper bound required: $\Phi^{min}_{i+1/2} \leq \Phi_{i+1/2} \leq \Phi^{max}_{i+1/2}$

$$\Delta\Phi_{i+1/2} = \text{sign}([\Delta\Phi_{i+1/2}]_{ave}) \cdot \min(|[\Delta\Phi_{i+1/2}]_{ave}|, 2|\Phi_{i+1/2} - \Phi^{min}_{i+1/2}|, 2|\Phi_{i+1/2} - \Phi^{max}_{i+1/2}|)$$

- harmonic mean:

$$[\Delta\Phi_{i+1/2}]_{mono4} = \begin{cases} \dfrac{\delta\Phi_i\delta\Phi_{i+1}}{[\delta\Phi_{i+1/2}]_{ave}} & \text{if sign}(\delta\Phi_i) = \text{sign}(\delta\Phi_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

- modified local lower/upper bound: $\Phi^{min}_{i+1/2} \leq \Phi_{i+1/2} \leq \Phi^{max}_{i+1/2}$

$$\Phi^{min}_{i+1/2} = \min(\Phi_{i-1/2}, \Phi_{i+1/2}, \Phi_{i+3/2}), \quad \Phi^{max}_{i+1/2} = \max(\Phi_{i-1/2}, \Phi_{i+1/2}, \Phi_{i+3/2})$$

- global lower and upper bound:

$$\Phi^{min}_{i+1/2} = 0, \quad \Phi^{max}_{i+1/2} = 1, \quad \text{for all } i$$

Mismatch and Monotonicity-Preserving

- harmonic mean:

$$[\Delta\Phi_{i+1/2}]_{mono4} = \begin{cases} \dfrac{\delta\Phi_i \delta\Phi_{i+1}}{[\delta\Phi_{i+1/2}]_{ave}} & \text{if sign}(\delta\Phi_i) = \text{sign}(\delta\Phi_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

- modified local lower/upper bound:

$$\Phi^{min}_{i+1/2} \leq \Phi_{i+1/2} \leq \Phi^{max}_{i+1/2}$$

$$\Phi^{min}_{i+1/2} = \min(\Phi_{i-1/2}, \Phi_{i+1/2}, \Phi_{i+3/2}),$$

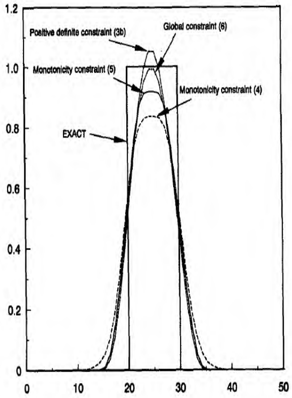
$$\Phi^{max}_{i+1/2} = \max(\Phi_{i-1/2}, \Phi_{i+1/2}, \Phi_{i+3/2})$$

- global lower and upper bound:

$$\Phi^{min}_{i+1/2} = 0, \quad \Phi^{max}_{i+1/2} = 1, \quad \text{for all } i$$

- positive definite:

$$\Phi^{min}_{1+1/2} = 0$$



Monotonicity preserving spectrum:

$$|[\Delta\Phi_{i+1/2}]_{ave}| \geq |[\Delta\Phi_{i+1/2}]_{posd}| \geq |[\Delta\Phi_{i+1/2}]_{min/max}| \geq |[\Delta\Phi_{i+1/2}]_{har}|$$

Outline

1. Basic Concepts of FVM
   - Monotonicity/TVD/L_1-Contracting/monotone methods
2. van-Leer Type Methods
   - van-Leer method with different mismatch values
3. 2/3D Problem and Splitting Methods
   - Splitting methods

Splitting Methods

Consider the differential equation:

$$\frac{\partial u}{\partial t} = (P_1 + P_2)u$$

where P_1 and P_2 are linear differential operators in spaces, for instance,

$$P_1 = -a_x \frac{\partial}{\partial x}, \quad P_2 = -a_y \frac{\partial}{\partial y}$$

Let Q_1 and Q_2 be approximate solvers for each part:

$$v^{n+1} = Q_1(k, t_n)v^n \qquad \text{approximating} \qquad \frac{\partial v}{\partial t} = P_1 v$$

$$w^{n+1} = Q_2(k, t_n)w^n \qquad \text{approximating} \qquad \frac{\partial w}{\partial t} = P_2 w$$

Then

$$u^{n+1} = Q_2 Q_1 u^n \qquad \frac{\partial u}{\partial t} = (P_1 + P_2)u \qquad \text{1st-order accurate}$$

$$u^{n+1} = Q_1(\frac{k}{2}, t_{n+1/2})Q_2(k, t_n)Q_1(\frac{k}{2}, t_n)u^n \qquad \frac{\partial u}{\partial t} = (P_1 + P_2)u \qquad \text{2nd-order accurate}$$

Splitting Methods

Consider the differential equation and the approximation methods:

$$\frac{\partial u}{\partial t} = (P_1 + P_2)u, \quad \begin{cases} v^{n+1} = Q_1(k, t_n)v^n & \text{approximating} \quad \dfrac{\partial v}{\partial t} = P_1 v \\ w^{n+1} = Q_2(k, t_n)w^n & \text{approximating} \quad \dfrac{\partial w}{\partial t} = P_2 w \end{cases}$$

Then

$$u^{n+1} = Q_2 Q_1 u^n \qquad \text{1st-order accurate}$$

$$u^{n+1} = Q_1\left(\frac{k}{2}, t_{n+1/2}\right) Q_2(k, t_n) Q_1\left(\frac{k}{2}, t_n\right) u^n \qquad \text{2nd-order accurate}$$

- The splitting method can be applied to nonlinear problem:

$$\frac{\partial \rho(x,y,t)}{\partial t} + \frac{\partial F(x,y,t)}{\partial x} + \frac{\partial G(x,y,t)}{\partial x} = 0.$$

- The splitting method can be applied to time-independent coefficient problem

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{d} P_j u, \quad v^{n+1} = Q_d Q_{d-1} \cdot Q_1 v^n \quad \text{accuracy } \mathcal{O}(k),$$

Note 2nd-order version does not generalize in a straightforward way.

- The accuracy of splitting methods used for problems with discontinuous solutions is not well understood.

References

- A class of the van Leer-type transport schemes and its application to the moisture transport in a general circulation model,
  S.-J. Lin, W. C. Chao, and G. K. Walker, Mon. Wea. Rev. (1994), 122, 1575-1593.
  - transport equation, van Leer/splitting method, sphere
- Multidimensional flux-form semi-Lagrangian transport schemes,
  S.-J. Lin and R. Rood, Mon. Wea. Rev. (1996), 124, 2046-2070.
  - transport equation, van Leer/splitting/semi-Lagrangian (flux-form), sphere
- An explicit flux-form semi-Lagrangian shallow-water model on the sphere,
  S.-J. Lin and R. Rood, Q. J. R. Meteorol. Soc. (1997), 123, 2477-2498.
  - shallow-water equation, van Leer/splitting/semi-Lagrangian (flux-form), sphere
- A "vertically Lagrangian" finite-volume dynamical core for global models,
  S.-J. Lin, Mon. Wea. Rev. (2004), 132, 2293-2307.
  - shallow-water equations, van Leer/splitting/ (horizontal/vertical) semi-Lagrangian (flux-form), sphere
- Finite-volume transport on various cubed-sphere grids,
  W. M. Putman and S. J. Lin, J. Comput. Phys. 2007, 227, 55-78.
  - transport equation, van Leer/splitting/ (horizontal) semi-Lagrangian (flux-form), cubed-sphere
- A two-way nested global-regional dynamical core on the cubed-sphere grid
  L. M. Harris, S.-J. Lin, Mon. Wea. Rev. (2013), 141, 283-306.
  - shallow-water equations, van Leer/splitting/ (horizontal/vertical) semi-Lagrangian (flux-form), grid-nesting, cubed-sphere